Research article

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On a degenerate hyperbolic problem for the 3-D steady full Euler equations with axial-symmetry

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Abstract: The transonic channel flow problem is one of the most important problems in mathematical fluid dynamics. The structure of solutions near the sonic curve is a key part of the whole transonic flow problem. This paper constructs a local classical hyperbolic solution for the 3-D axisymmetric steady compressible full Euler equations with boundary data given on the degenerate hyperbolic curve. By introducing a novel set of dependent and independent variables, we use the idea of characteristic decomposition to transform the axisymmetric Euler equations as a new system which has explicitly singularity-regularity structures. We first establish a local classical solution for the new system in a weighted metric space and then convert the solution in terms of the original variables.

Keywords: Full Euler equations, axial-symmetry, degenerate hyperbolic, classical solution, characteristic decomposition

MSC: 35L65, 35L80, 76H05

1 Introduction

The three-dimensional steady compressible full Euler equations read that [12]

$$\begin{cases} (\rho u_1)_x + (\rho u_2)_y + (\rho u_3)_z = 0, \\ (\rho u_1^2 + p)_x + (\rho u_1 u_2)_y + (\rho u_1 u_3)_z = 0, \\ (\rho u_1 u_2)_x + (\rho u_2^2 + p)_y + (\rho u_2 u_3)_z = 0, \\ (\rho u_1 u_3)_x + (\rho u_2 u_3)_y + (\rho u_3^2 + p)_z = 0, \\ (\rho E u_1 + p u_1)_x + (\rho E u_2 + p u_2)_y + (\rho E u_3 + p u_3)_z = 0, \end{cases} \tag{1.1}$$

where ρ, (u_1, u_2, u_3), p and E are, respectively, the density, the velocity, the pressure and the specific total energy. For polytropic gases, $E = \frac{u_1^2+u_2^2+u_3^2}{2} + \frac{p}{(\gamma-1)\rho}$, where $\gamma > 1$ is the adiabatic gas constant.

In this paper, we are focused on the three-dimensions steady compressible flows with axial symmetry in gas dynamics. This kind of motions arise naturally in many physical and engineering situations, for example, the flow through an axisymmetric nozzle, see [2, 12] and references cited therein. Taking the x-axis to be the axis of symmetry and setting $(r, \sigma) = (\sqrt{y^2 + z^2}, \arctan(z/y))$ to be the polar coordinates of y-z plane, then the flow is axially symmetric in mathematically if its state is dependent only on (x, r) but independent of σ. Assume that the flow is axially symmetric, we let the fluid density, velocity and pressure be $\rho(x, r)$,

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$(u, v, w)(x, r)$ and $p(x, r)$ in cylindrical coordinates (x, r, σ), where u, v, w are, respectively, the axial velocity, radial velocity and swirl velocity, that is,

$$u_1 = u, \quad \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

Then system (1.1) reduces to

$$\begin{cases} (\rho u)_x + (\rho v)_r = -\frac{\rho v}{r}, \\ (\rho u^2 + p)_x + (\rho u v)_r = -\frac{\rho u v}{r}, \\ (\rho u v)_x + (\rho v^2 + p)_r = -\frac{\rho(v^2 - w^2)}{r}, \\ (\rho u w)_x + (\rho v w)_r = -\frac{2\rho v w}{r}, \\ (\rho u E + p u)_x + (\rho v E + p v)_r = -\frac{\rho v E + p v}{r}. \end{cases} \tag{1.2}$$

The eigenvalues of system (1.2) are

$$\Lambda_0 = \Lambda_1 = \Lambda_2 = \frac{v}{u}, \quad \Lambda_\pm = \frac{uv \pm c\sqrt{q^2 - c^2}}{u^2 - c^2}, \tag{1.3}$$

where $c = \sqrt{\gamma p/\rho}$ is the sound speed and $q = \sqrt{u^2 + v^2}$. It is clear that system (1.2) is hyperbolic for $q > c$ and degenerate hyperbolic for $q = c$. We mention that the point at which $q = c$ is not the sonic point unless the swirl velocity $w = 0$.

This paper aims to explore the structure of solutions near the degenerate curves for the 3-D axisymmetric steady full Euler equations (1.2). The motivation for studying such problem comes from the channel flow problems which contain flows over airfoils or in nozzles and have been studied for a long time. Many pieces of work have been contributed on the steady isentropic irrotational plane flows due to the fact that the control system can be transformed into a two-by-two quasilinear reducible equations or a single second order nonlinear (potential) equation, see, e.g., [1, 4, 20, 23, 34, 35, 37]. For an axially symmetric flow, Gilbarg [21] used the comparison principle to verify that the flow speed on the boundary is increasing with respect to the incoming mass flux if the flow approximates to uniform flow at far fields. An uniqueness result for the axially symmetric subsonic flow past a body was given by Gilbarg and Serrin [22]. In the past few years, the existence of global solutions in a subsonic-sonic part of the nozzle have been intensively investigated. The existence and stability of multidimensional transonic potential flows through an infinite nozzle of arbitrary cross-sections was established by Chen and Feldman [7]. In [43, 44], Xie and Xin established the existence of global subsonic-sonic solutions for 2-D and 3-D axially symmetric nozzles. Du, Xin and Yan [17] discussed the existence and uniqueness of global subsonic solutions for the multidimensional potential equations. The well-posedness for the 2-D subsonic and subsonic-sonic flows with critical mass flux was established by Xie and Xin [45] for the isentropic Euler equations and by Chen, Deng and Xiang [5] for the full Euler equations, also see Du and Duan [14, 15] and Duan and Luo [18] for the corresponding results to the 3-D axisymmetric cases. In [40, 42], Wang and Xin established the existence and uniqueness of smooth transonic flows in Laval nozzles for the potential equation. The existence of the 3-D axisymmetric Euler flows through piecewise smooth nozzles with nontrivial swirl was recently studied by Deng, Wang and Xiang [13]. For more related results, one may refer to [3, 8, 16, 39, 41] and references therein. One may also see [6, 9–11, 19, 46] for the results of transonic shocks arising in supersonic flow past a blunt body or a bounded nozzle. On the other hand, a local sonic-supersonic classical solution for the 2-D steady isentropic irrotational Euler equations was constructed by Zhang and Zheng [47]. This sonic-supersonic solution may be expected to combine with the subsonic-sonic solution of Xie and Xin [43] by using an iterative process. In [24], Hu and Li verified the existence of classical sonic-supersonic solutions for the 2-D steady full Euler equations. The relevant researches of the 2-D pseudo-steady Euler system were provided in [48] for the isentropic irrotational case and in [27] for the full case.

In the present paper, we establish the local existence of classical solutions to the 3-D axisymmetric full Euler system (1.2) with degenerate boundary data in the hyperbolic region. To our best knowledge, this is the first time to construct a classical hyperbolic solution near a degenerate curve for the 3-D axisymmetric

Euler equations, especially for the case with non-zero swirl component. We expect that the result of the paper contributes to not only clarify the singularity structure of solutions at a degenerate hyperbolic boundary but also understand the effect of non-zero swirl velocity on the degeneracy for the 3-D compressible Euler system. We consider the problem as follows.

Problem 1.1. *Let $\Gamma : r = \varphi(x)$, $x \in [x_1, x_2]$ be a smooth curve satisfying $0 < r_1 \le \varphi(x) \le r_2$ for any $x \in [x_1, x_2]$, where r_1 and r_2 are two positive constants. We assign the boundary data for (ρ, u, v, w, p) on Γ, $(\rho, u, v, w, p)(x, \varphi(x)) = (\hat{\rho}, \hat{u}, \hat{v}, \hat{w}, \hat{p})(x)$ such that $\hat{\rho}(x) > 0$, $\hat{p}(x) > 0$ and $\hat{u}(x)^2 + \hat{v}(x)^2 = \gamma\hat{p}(x)/\hat{\rho}(x)$ for any $x \in [x_1, x_2]$. This means that system* (1.2) *is hyperbolic degenerate at the curve Γ. We look for a classical solution for* (1.2) *in the region $q > c$ near Γ.*

The main difficulty solving Problem 1.1 arises from the singularities caused by the degeneracy of system (1.2) on Γ. To overcome it, the key strategy is to isolate the singular terms in the highly complicated full Euler system. It is decisive to transform the Euler equations as a new and effective governing system by choosing a pair of appropriate independent coordinates. For the isentropic irrotational plane flows, the velocity functions were usually chosen as the independent variables to linearize the Euler equations, which is the well-known hodograph method. However, this method is difficult in taking on boundary conditions and in returning back to the original variables due to the degeneracy. Kuzmin [28] took the stream and potential functions as the coordinate plane to study the transonic perturbation problems. Moreover, Zhang and Zheng's previous result [47] and other related works [38, 48] were based on the coordinate system composed by the variable $\sqrt{q^2 - c^2}$ and the potential function. Due to the non-existence of potential function, it seems impossible to apply the above coordinate systems to deal with the full Euler equations. In [24], the use of the angle functions as the independent variables was proposed by Hu and Li. It turns out that the angle coordinate system works well for the full Euler system, see [25, 27] for more applications. In the current paper, we introduce the angle variables for the 3-D axisymmetric full Euler equations (1.2) as the independent coordinate functions to obtain a new system which displays a clear regularity-singularity structure.

The main result in this paper is stated in the following theorem.

Theorem 1.1. *Let $\hat{\theta}$ be an angle function on Γ defined by $\hat{\theta} = \arctan(\hat{v}/\hat{u})$. Suppose that the curve Γ and the boundary data $(\hat{\rho}, \hat{u}, \hat{v}, \hat{w}, \hat{p})$ satisfy*

$$\begin{aligned}&(A_1): \varphi(x) \in C^4([x_1, x_2]), \quad (\hat{\rho}, \hat{u}, \hat{v}, \hat{w}, \hat{p})(x) \in C^4([x_1, x_2]),\\&(A_2): \varphi' \cos\hat{\theta} - \sin\hat{\theta} > 0, \quad \cos\hat{\theta} + \varphi' \sin\hat{\theta} > 0,\\&(A_3): \hat{p}' \le 0, \quad \hat{\theta}' < -\tfrac{\sin\hat{\theta}(\varphi' \cos\hat{\theta} - \sin\hat{\theta})}{\varphi(x)} \le 0, \quad \forall\, x \in [x_1, x_2].\end{aligned} \tag{1.4}$$

Then there exists a classical solution for Problem 1.1 in the region $q > c$ near Γ.

Remark 1. *The regularity conditions (A_1) may be relaxed somewhat by using the concept of modulus of continuity. The conditions (A_2) mean that the direction $(\cos\theta, \sin\theta)$ is neither tangent nor normal to the curve Γ. The conditions (A_2) and (A_3) are mainly used to ensure that the variable q/c is strictly increasing along the flow direction $(\cos\theta, \sin\theta)$ on the curve Γ. This requirement is reasonable since we expect to construct a classical solution from the curve at which $q/c = 1$ to the region $q/c > 1$. When the initial swirl velocity $\hat{w} \ne 0$, the condition (A_3) can be relaxed, which means that the swirl component plays a positive role in this degenerate problem, see Remark 2 in Section 2.*

To prove Theorem 1.1, we introduce the Mach-like angle ω and the flow-like angle θ and then choose $(\cos\omega, \theta)$ as the independent coordinate system. To get a new suitable governing system, we further introduce a novel variable Ξ, a function of Mach-like angle ω, entropy S and Bernoulli-like quantity B, and then derive the characteristic decompositions of Ξ under a pair of new weighted directional derivatives $(\tilde{\partial}^+, \tilde{\partial}^-)$, see Section 2 below. With the help of technical characteristic decompositions, the axisymmetric Euler equations (1.2) can be transformed to a new closed system in the partial hodograph $(\cos\omega, \theta)$-plane. The new system has the desired explicitly singularity-regularity structures, although its expressions become considerably more complex. We establish the local existence of classical solutions for the new system in a weighted metric space by the fixed-point method. Going back to the original variables, we thus obtain a local classical hyperbolic solution to (1.2) with degenerate boundary data.

The rest of the paper is organized as follows. In Section 2, we introduce the angle functions as the dependent variables and then reformulate the problem in terms of the angle variables by deriving their characteristic decompositions. In Section 3, we solve the local existence problem in a partial hodograph coordinate plane by using the iteration method in a weighted metric space. In Section 4, we convert the classical solution in the partial hodograph plane to that in the original physical plane to complete the proof of the main theorem. Finally, we provide the derivations of some complicated formulas in appendices.

2 Formulation of the problem in angle variables

In this section, we introduce the angle functions as the dependent variables and derive their characteristic decompositions to restate the problem in a new framework. The angle functions as the dependent variables was invented by Li and Zheng [31] and then was applied in many other problems, see, e.g. [26, 29, 30, 32, 33, 36].

2.1 Characteristic decompositions of angle variables

For smooth solutions, we can write system (1.2) as

$$\mathbf{A}\mathbf{W}_x + \mathbf{B}\mathbf{W}_r = \mathbf{C}, \tag{2.1}$$

where

$$\mathbf{W} = \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \frac{u}{\rho} & 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & u \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \frac{v}{\rho} & 0 & 1 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \gamma p & 0 & v \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -\frac{v}{r} \\ 0 \\ \frac{w^2}{r} \\ -\frac{wv}{r} \\ -\frac{\gamma p v}{r} \end{pmatrix}.$$

The five eigenvalues of (2.1) are given in (1.3), and the corresponding five left eigenvectors are

$$\ell_0 = (-\gamma p, 0, 0, 0, 1),\ \ell_1 = (0, u, v, 0, 0),\ \ell_2 = (0, 0, 0, 1, 0),\ \ell_\pm = (0, \Lambda_\pm \gamma p, -\gamma p, 0, v - \Lambda_\pm u).$$

By a direct calculation, one obtains the characteristic form of system (2.1)

$$\begin{cases} uS_x + vS_r = 0, \\ uB_x + vB_r = \frac{vw^2}{r}, \\ uw_x + vw_r = -\frac{vw}{r}, \\ \gamma pv(u_x + \Lambda_+ u_r) - \gamma pu(v_x + \Lambda_+ v_r) - c\sqrt{q^2 - c^2}(p_x + \Lambda_+ p_r) = -\frac{\gamma p(w^2 + v^2 - uv\Lambda_+)}{r}, \\ \gamma pv(u_x + \Lambda_- u_r) - \gamma pu(v_x + \Lambda_- v_r) + c\sqrt{q^2 - c^2}(p_x + \Lambda_- p_r) = -\frac{\gamma p(w^2 + v^2 - uv\Lambda_-)}{r}, \end{cases} \tag{2.2}$$

where $c = \sqrt{\gamma p/\rho}$, $q = \sqrt{u^2 + v^2}$, $S = p\rho^{-\gamma}$ and $B = \frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1}$. We call S the entropy function and B the Bernoulli-like function. The equation for B in (2.2) can be replaceable by the following equation

$$u\tilde{B}_x + v\tilde{B}_r = 0,$$

where $\tilde{B} = \frac{u^2+v^2+w^2}{2} + \frac{c^2}{\gamma-1}$ is the Bernoulli function. The reason we are not using the variable $\tilde{B}$ is that it will lead the last two equations in (2.2) to be more complicated in later calculations.

We introduce the flow-like angle θ and the Mach-like angle ω as follows

$$\tan\theta = \frac{v}{u}, \quad \sin\omega = \frac{c}{q}, \tag{2.3}$$

and denote

$$\alpha := \theta + \omega, \quad \beta := \theta - \omega. \tag{2.4}$$

It is easily checked by the expressions of eigenvalues in (1.3) that

$$\tan\alpha = \Lambda_+, \quad \tan\beta = \Lambda_-, \quad \tan\theta = \Lambda_0 = \Lambda_1 = \Lambda_2, \tag{2.5}$$

which indicates that the angles α, β, θ are the inclination angles of characteristic curves. Moreover, we can use the angle variables to express the functions u, v and c

$$u = c\frac{\cos\theta}{\sin\omega}, \quad v = c\frac{\sin\theta}{\sin\omega}, \quad c = \frac{\sqrt{2\kappa B}\sin\omega}{\sqrt{\kappa + \sin^2\omega}}, \tag{2.6}$$

where $\kappa = (\gamma - 1)/2$. Furthermore, we define the following weighted directional derivatives along the characteristics

$$\begin{aligned} \tilde{\partial}^+ &= r\cos\alpha\partial_x + r\sin\alpha\partial_r, \quad \tilde{\partial}^- = r\cos\beta\partial_x + r\sin\beta\partial_r,\\ \tilde{\partial}^0 &= r\cos\theta\partial_x + r\sin\theta\partial_r, \quad \tilde{\partial}^\perp = r\sin\theta\partial_x - r\cos\theta\partial_r, \end{aligned} \tag{2.7}$$

from which one obtains

$$\begin{cases} \partial_x = -\frac{\sin\beta\tilde{\partial}^+ - \sin\alpha\tilde{\partial}^-}{r\sin(2\omega)},\\ \partial_r = \frac{\cos\beta\tilde{\partial}^+ - \cos\alpha\tilde{\partial}^-}{r\sin(2\omega)}, \end{cases} \qquad \begin{cases} \tilde{\partial}^0 = \frac{\tilde{\partial}^+ + \tilde{\partial}^-}{2\cos\omega},\\ \tilde{\partial}^\perp = \frac{\tilde{\partial}^- - \tilde{\partial}^+}{2\sin\omega}. \end{cases} \tag{2.8}$$

From (2.2) and (2.7), we get a new system in terms of the variables $(S, B, w, \omega, \theta)$

$$\begin{cases} \tilde{\partial}^0 S = 0,\\ \tilde{\partial}^0 B = w^2\sin\theta,\\ \tilde{\partial}^0 w = -w\sin\theta,\\ \tilde{\partial}^+\theta + \frac{\cos^2\omega}{\kappa+\sin^2\omega}\tilde{\partial}^+\omega - \frac{\sin(2\omega)}{4\kappa}\tilde{\partial}^+\big(\frac{1}{\gamma}\ln S - \ln B\big) = G(\kappa + \sin^2\omega)\cos\alpha - \sin\omega\sin\theta,\\ \tilde{\partial}^-\theta - \frac{\cos^2\omega}{\kappa+\sin^2\omega}\tilde{\partial}^-\omega + \frac{\sin(2\omega)}{4\kappa}\tilde{\partial}^-\big(\frac{1}{\gamma}\ln S - \ln B\big) = G(\kappa + \sin^2\omega)\cos\beta + \sin\omega\sin\theta. \end{cases} \tag{2.9}$$

The detailed derivation of (2.9) is presented in Appendix A. Here $G = w^2/(2\kappa B)$ satisfies the following equation

$$\tilde{\partial}^0 G = -2(1 + \kappa G)G\sin\theta, \tag{2.10}$$

which is derived from the second and third equations in (2.9).

In order to derive the characteristic decompositions for the angle variables, we introduce a new variable

$$\Xi = \frac{1}{4\kappa}\ln\left(\frac{\sin^2\omega}{\kappa + \sin^2\omega}\right) - \frac{1}{4\kappa}\left(\frac{1}{\gamma}\ln S - \ln B\right). \tag{2.11}$$

Obviously, the last two equations of (2.9) reduce to

$$\begin{cases} \tilde{\partial}^+\theta + \sin(2\omega)\tilde{\partial}^+\Xi = G(\kappa + \sin^2\omega)\cos\alpha - \sin\omega\sin\theta,\\ \tilde{\partial}^-\theta - \sin(2\omega)\tilde{\partial}^-\Xi = G(\kappa + \sin^2\omega)\cos\beta + \sin\omega\sin\theta. \end{cases} \tag{2.12}$$

To acquire a closed hyperbolic system, we adopt the weighted directional derivatives of (Ξ, S, B, G) as the dependent variables. Denote

$$X = \tilde{\partial}^+\Xi, \ Y = \tilde{\partial}^-\Xi, \ F = \tilde{\partial}^+ G, \ H = \tilde{\partial}^+\big(\frac{1}{4\kappa\gamma}\ln S - \frac{1}{4\kappa}\ln B\big). \tag{2.13}$$

Taking advantage of (2.8) and (2.9) yields

$$\tilde{\partial}^-\left(\frac{1}{4\kappa\gamma}\ln S - \frac{1}{4\kappa}\ln B\right) = -H - G\cos\omega\sin\theta. \tag{2.14}$$

In addition, by the definition of Ξ, one has

$$\begin{aligned}\tilde{\partial}^{+}\omega &= \frac{2\sin\omega(\kappa+\sin^2\omega)}{\cos\omega}(X+H),\\ \tilde{\partial}^{-}\omega &= \frac{2\sin\omega(\kappa+\sin^2\omega)}{\cos\omega}(Y-H-G\cos\omega\sin\theta).\end{aligned}\tag{2.15}$$

On the other hand, we have the following commutator relations of second derivative by performing a direct calculation

$$\begin{aligned}\tilde{\partial}^{-}\tilde{\partial}^{+}-\tilde{\partial}^{+}\tilde{\partial}^{-} &= \frac{\cos(2\omega)\tilde{\partial}^{-}\alpha-\tilde{\partial}^{+}\beta}{\sin(2\omega)}\tilde{\partial}^{+}+\frac{\cos(2\omega)\tilde{\partial}^{+}\beta-\tilde{\partial}^{-}\alpha}{\sin(2\omega)}\tilde{\partial}^{-}+\sin\beta\tilde{\partial}^{+}-\sin\alpha\tilde{\partial}^{-},\\ \tilde{\partial}^{0}\tilde{\partial}^{+}-\tilde{\partial}^{+}\tilde{\partial}^{0} &= \frac{\cos\omega\tilde{\partial}^{0}\alpha-\tilde{\partial}^{+}\theta}{\sin\omega}\tilde{\partial}^{+}+\frac{\cos\omega\tilde{\partial}^{+}\theta-\tilde{\partial}^{0}\alpha}{\sin\omega}\tilde{\partial}^{0}+\sin\theta\tilde{\partial}^{+}-\sin\alpha\tilde{\partial}^{0}.\end{aligned}\tag{2.16}$$

Combining with (2.9), (2.10), (2.12), (2.13), (2.14) and (2.16), we can obtain a closed system in terms of the variables (G, H, F, X, Y)

$$\begin{cases}\tilde{\partial}^{+}G = F,\\ \tilde{\partial}^{0}H = \frac{G(k+\sin^2\omega)}{\cos\omega}\left(\sin\theta\frac{X+Y}{2\cos\omega}-\frac{G\cos^2\alpha}{2}\right)+\frac{(\kappa+1)(X+Y)+2\sin\theta\cos\omega}{\cos\omega}H\\ \qquad +\frac{G\sin(2\omega+\theta)}{2}X+\frac{G\sin\theta}{2}Y-\frac{\sin\theta}{2}F+G\sin\theta\sin\alpha,\\ \tilde{\partial}^{0}F = \frac{2G(1+\kappa G)(\kappa+\sin^2\omega)}{\cos\omega}\left(\sin\theta\frac{X+Y}{\cos\omega}-G\cos^2\alpha\right)+\frac{(\kappa+1)(X+Y)-4\kappa G\sin\theta\cos\omega}{\cos\omega}F\\ \qquad +2G(1+\kappa G)[\sin(2\omega+\theta)X+\sin\theta Y+2\sin\theta\sin\alpha],\\ \tilde{\partial}^{-}X = (\kappa+\sin^2\omega)(X+H)\frac{X+Y}{\cos^2\omega}+X[\cos(2\omega)Y+X+2\sin\theta\cos\omega+\frac{\sin\beta}{2}]\\ \qquad +Y[\frac{\sin\beta}{2}-2(\kappa+\sin^2\omega)(X+H)]+\sin^2\theta+G\sin\beta(\kappa+\sin^2\omega)X+\Phi\\ \tilde{\partial}^{+}Y = (\kappa+\sin^2\omega)(Y-H)\frac{X+Y}{\cos^2\omega}+Y[\cos(2\omega)X+Y+2\sin\theta\cos\omega+\frac{\sin\alpha}{2}]\\ \qquad +X[\frac{\sin\alpha}{2}-2(\kappa+\sin^2\omega)(Y-H)]+\sin^2\theta+G\sin\alpha(\kappa+\sin^2\omega)Y+\Phi\\ \qquad -G\sin\theta(\kappa+\sin^2\omega)\frac{X+Y}{\cos\omega}+G\sin\theta(\kappa+\sin^2\omega)\cos\omega X,\end{cases}\tag{2.17}$$

where

$$\begin{aligned}\Phi = &-G\sin^2\omega\sin\theta(\kappa+\sin^2\omega)\frac{X+Y}{\cos\omega}+G(Y-X)\sin\omega\cos\theta(\kappa+\sin^2\omega)\\ &-2GH\sin\omega\cos\theta(\kappa+\sin^2\omega)-G^2\frac{(\kappa+\sin^2\omega)^2\sin\theta\cos\alpha}{\sin\omega}\\ &+G\frac{\sin\theta(\kappa+\sin^2\omega)}{\sin\omega}(2\sin\theta\sin\omega-\cos\theta\cos\omega)-F\frac{(\kappa+\sin^2\omega)\cos\theta}{2\sin\omega}.\end{aligned}\tag{2.18}$$

The detailed derivation of (2.17) is given in Appendix B. If we get (G, H, F, X, Y) by solving (2.17), we can obtain θ and ω from (2.12) and (2.15) and then solve the first three equations of (2.9) to arrive at the functions S, B and w.

2.2 The boundary data and restatement of the result

Corresponding to the boundary data (1.4), we now derive the the boundary condition for system (2.17).

By the definitions of S, B and G, we first have

$$\begin{aligned}&\hat{S}(x)=\hat{p}(x)\hat{\rho}^{-\gamma}(x)>0,\quad \hat{G}(x)=\frac{\hat{\rho}(x)\hat{w}^2(x)}{(\kappa+1)\gamma\hat{p}(x)}\geq 0,\\ &\hat{B}(x)=\frac{\hat{u}^2(x)+\hat{v}^2(x)}{2}+\frac{\gamma\hat{p}(x)}{(\gamma-1)\hat{\rho}(x)}=\frac{(\kappa+1)\gamma\hat{p}(x)}{2\kappa\hat{\rho}(x)}>0,\quad \forall\, x\in[x_1,x_2].\end{aligned}\tag{2.19}$$

We next the boundary value of H on Γ. It follows by the definition of H and (2.19) that

$$H|_{\Gamma}=\hat{H}(x):=\left(\frac{2\kappa}{\gamma(\kappa+1)}\right)^{-\gamma}\frac{\hat{p}^{\gamma-1}}{4\kappa\gamma}\cdot\tilde{\partial}^{+}\left(\frac{S}{B^{\gamma}}\right)\bigg|_{\Gamma}.\tag{2.20}$$

Recalling $S(x,\varphi(x))=\hat{S}(x)$ on Γ and applying the equation $\tilde{\partial}^0 S=0$ gives

$$S_x(x,\varphi(x))=-\frac{\hat{S}'\sin\hat{\theta}}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}},\quad S_r(x,\varphi(x))=\frac{\hat{S}'\cos\hat{\theta}}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}},$$

from which one has

$$\tilde{\partial}^+ S|_\Gamma=\frac{\varphi\hat{S}'}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}. \tag{2.21}$$

Similarly, we use the equation $\tilde{\partial}^0 B=w^2\sin\theta$ on Γ to arrive at

$$B_x(x,\varphi(x))=-\frac{\varphi\hat{B}'\sin\hat{\theta}-w^2\varphi'\sin\hat{\theta}}{\varphi(\varphi'\cos\hat{\theta}-\sin\hat{\theta})},\quad B_r(x,\varphi(x))=\frac{\varphi\hat{B}'\cos\hat{\theta}-w^2\sin\hat{\theta}}{\varphi(\varphi'\cos\hat{\theta}-\sin\hat{\theta})}.$$

Thus we have

$$\tilde{\partial}^+ B|_\Gamma=\frac{\varphi\hat{B}'-w^2\sin\hat{\theta}(\varphi'\sin\hat{\theta}+\cos\hat{\theta})}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}. \tag{2.22}$$

Inserting (2.21) and (2.22) into (2.20) and doing a simple calculation yields

$$H|_\Gamma=\hat{H}(x)=-\frac{\varphi\hat{p}'}{2\gamma\hat{p}(\varphi'\cos\hat{\theta}-\sin\hat{\theta})}+\frac{(\varphi'\sin\hat{\theta}+\cos\hat{\theta})\sin\hat{\theta}}{2(\varphi'\cos\hat{\theta}-\sin\hat{\theta})}\hat{G}. \tag{2.23}$$

For the boundary value of F, we see that

$$F|_\Gamma=\hat{F}(x):=\tilde{\partial}^+ G|_\Gamma=\tilde{\partial}^+\left(\frac{w^2}{2\kappa B}\right)\bigg|_\Gamma=\frac{\hat{w}}{\kappa\hat{B}}\tilde{\partial}^+ w|_\Gamma-\frac{\hat{w}^2}{2\kappa\hat{B}^2}\tilde{\partial}^+ B|_\Gamma. \tag{2.24}$$

We recall the equation $\tilde{\partial}^0 w=-w\sin\theta$ in (2.9) and the function $w(x,\varphi(x))=\hat{w}(x)$ on Γ to find that

$$w_x(x,\varphi(x))=-\frac{\varphi\hat{w}'\sin\hat{\theta}+\varphi'\hat{w}\sin\hat{\theta}}{\varphi(\varphi'\cos\hat{\theta}-\sin\hat{\theta})},\quad w_r(x,\varphi(x))=\frac{\varphi\hat{w}'\cos\hat{\theta}+\hat{w}\sin\hat{\theta}}{\varphi(\varphi'\cos\hat{\theta}-\sin\hat{\theta})},$$

which together with the fact $\omega=\pi/2$ on Γ lead to

$$\tilde{\partial}^+ w|_\Gamma=\frac{\varphi\hat{w}'+\hat{w}\sin\hat{\theta}(\varphi'\sin\hat{\theta}+\cos\hat{\theta})}{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}.$$

Putting the above into (2.24) and employing (2.22) gets

$$F|_\Gamma=\hat{F}(x)=\frac{\varphi\hat{w}(2\hat{B}\hat{w}'-\hat{B}'\hat{w})+\hat{w}^2\sin\hat{\theta}(\varphi'\sin\hat{\theta}+\cos\hat{\theta})(2\hat{B}+\hat{w}^2)}{2\kappa\hat{B}^2(\varphi'\cos\hat{\theta}-\sin\hat{\theta})}. \tag{2.25}$$

We now consider the boundary data (X,Y) on Γ. From (2.8) and (2.13), one has $X+Y=2\cos\omega\tilde{\partial}^0\Xi$ which implies that $X=-Y$ on Γ for smooth solutions. On the other hand, we add the two equations in (2.12) to obtain

$$X-Y=-\frac{\tilde{\partial}^+\theta+\tilde{\partial}^-\theta}{\sin(2\omega)}+\frac{(\kappa+\sin^2\omega)\cos\theta}{\sin\omega}G=-\frac{\tilde{\partial}^0\theta}{\sin\omega}+\frac{(\kappa+\sin^2\omega)\cos\theta}{\sin\omega}G,$$

which indicates that on Γ

$$X=-Y=-\frac{1}{2}\tilde{\partial}^0\theta|_\Gamma+\frac{(\kappa+1)\cos\hat{\theta}}{2}\hat{G}. \tag{2.26}$$

To acquire the value $\tilde{\partial}^0\theta$ on Γ, we subtract the two equations in (2.12) to derive

$$\tilde{\partial}^\perp\theta=(X+Y)\cos\omega+[G(\kappa+\sin^2\omega)+1)]\sin\theta,$$

which means that $\tilde{\partial}^{\perp}\theta|_{\Gamma} = [\hat{G}(\kappa+1)+1]\sin\hat{\theta}$. Hence we obtain

$$\theta_x(x,\varphi(x)) = \frac{\varphi\hat{\theta}'\cos\hat{\theta} + [\hat{G}(\kappa+1)+1]\varphi'\sin\hat{\theta}}{\varphi(\sin\hat{\theta}\varphi' + \cos\hat{\theta})},$$
$$\theta_r(x,\varphi(x)) = \frac{\varphi\hat{\theta}'\sin\hat{\theta} - [\hat{G}(\kappa+1)+1]\sin\hat{\theta}}{\varphi(\sin\hat{\theta}\varphi' + \cos\hat{\theta})},$$

from which one has

$$\tilde{\partial}^0\theta|_{\Gamma} = \frac{\varphi\hat{\theta}' + [\hat{G}(\kappa+1)+1](\varphi'\cos\hat{\theta} - \sin\hat{\theta})\sin\hat{\theta}}{\varphi'\sin\hat{\theta} + \cos\hat{\theta}}. \tag{2.27}$$

Combining with (2.26) and (2.27) gives

$$Y|_{\Gamma} = -X|_{\Gamma} = \frac{\varphi\hat{\theta}' + \sin\hat{\theta}(\varphi'\cos\hat{\theta} - \sin\hat{\theta}) - (\kappa+1)\hat{G}}{2(\varphi'\sin\hat{\theta} + \cos\hat{\theta})} =: \hat{a}_0(x). \tag{2.28}$$

We next check the value $\tilde{\partial}^0(q/c)$ on Γ for better understanding the conditions (A_2) and (A_3) in Theorem 1.1. According to the last two equations of (2.9), we find that

$$\tilde{\partial}^0\theta - \frac{\sin\omega}{\kappa+\sin^2\omega}\tilde{\partial}^{\perp}(\sin\omega) = -\frac{\sin^2\omega}{2\kappa}\tilde{\partial}^{\perp}\big(\frac{1}{\gamma}\ln S - \ln B\big) + (\kappa+\sin^2\omega)\cos\theta G,$$

which along with the facts $\bar{\partial}^+S + \bar{\partial}^-S = 0$, $\bar{\partial}^+B + \bar{\partial}^-B = 0$ on Γ suggests

$$\begin{aligned}\tilde{\partial}^{\perp}\sin\omega|_{\Gamma} =&(\kappa+1)\Big\{\tilde{\partial}^0\theta|_{\Gamma} + \frac{\hat{B}^{\gamma}}{2\kappa\gamma\hat{S}}\tilde{\partial}^{\perp}\Big(\frac{S}{B^{\gamma}}\Big)\Big|_{\Gamma} - (\kappa+1)\cos\hat{\theta}\hat{G}\Big\}\\ =&2(\kappa+1)[\hat{a}_0(x) - \hat{H}(x)].\end{aligned}$$

Here we have used the relation $\tilde{\partial}^{\perp}\Big(\frac{S}{B^{\gamma}}\Big) = -\bar{\partial}^+\Big(\frac{S}{B^{\gamma}}\Big)$.We note that fact $\sin\omega = 1$ on Γ to derive

$$(\partial_x\sin\omega)|_{\Gamma} = \frac{2\varphi'(\kappa+1)(\hat{a}_0(x) - \hat{H}(x))}{\varphi(\varphi'\sin\hat{\theta} + \cos\hat{\theta})}, \quad (\partial_r\sin\omega)|_{\Gamma} = -\frac{2(\kappa+1)(\hat{a}_0(x) - \hat{H}(x))}{\varphi(\varphi'\sin\hat{\theta} + \cos\hat{\theta})},$$

from which one has

$$\tilde{\partial}^0\sin\omega|_{\Gamma} = \frac{2(\kappa+1)(\varphi'\cos\hat{\theta} - \sin\hat{\theta})}{\varphi'\sin\hat{\theta} + \cos\hat{\theta}}[\hat{a}_0(x) - \hat{H}(x)]. \tag{2.29}$$

It follows that

$$\tilde{\partial}^0\Big(\frac{q}{c}\Big)\Big|_{\Gamma} = \tilde{\partial}^0\Big(\frac{1}{\sin\omega}\Big)\Big|_{\Gamma} = -(\tilde{\partial}^0\sin\omega)|_{\Gamma} = \frac{2(\kappa+1)(\varphi'\cos\hat{\theta} - \sin\hat{\theta})}{\varphi'\sin\hat{\theta} + \cos\hat{\theta}}[\hat{H}(x) - \hat{a}_0(x)].$$

Thus $\tilde{\partial}^0(q/c)|_{\Gamma}$ and $[\hat{H}(x) - \hat{a}_0(x)]$ have the equivalent symbol for all $x \in [x_1, x_2]$ by the assumption (A_2) in (1.4). Moreover, we recall (2.23) and (2.28) to compute

$$\begin{aligned}\hat{H}(x) - \hat{a}_0(x) =& -\frac{\varphi\hat{p}'}{2\gamma\hat{p}(\varphi'\cos\hat{\theta} - \sin\hat{\theta})} - \frac{\varphi\hat{\theta}' + \sin\hat{\theta}(\varphi'\cos\hat{\theta} - \sin\hat{\theta})}{2(\varphi'\sin\hat{\theta} + \cos\hat{\theta})}\\ &+ \frac{(\kappa+1)(\varphi'\cos\hat{\theta} - \sin\hat{\theta}) + \sin\hat{\theta}(\varphi'\sin\hat{\theta} + \cos\hat{\theta})^2}{2(\kappa+1)\gamma\hat{p}(\varphi'\cos\hat{\theta} - \sin\hat{\theta})(\varphi'\sin\hat{\theta} + \cos\hat{\theta})}\hat{\rho}\hat{w}^2.\end{aligned} \tag{2.30}$$

With the help of the assumptions (A_2) and (A_3) in (1.4), we see that $[\hat{H}(x) - \hat{a}_0(x)] > 0$ for all $x \in [x_1, x_2]$ and then $\tilde{\partial}^0(q/c) > 0$ on Γ.

We summarize (2.19), (2.23), (2.25) and (2.28) to get the boundary data (G, H, F, X, Y) on Γ with

$$(G, H, F, X, Y)|_{\Gamma} = (\hat{G}, \hat{H}, \hat{F}, -\hat{a}_0, \hat{a}_0)(x) \in C^3([x_1, x_2]). \tag{2.31}$$

Then Theorem 1.1 is restated in the next theorem.

Theorem 2.1. *Let assumptions (A_2) and (A_3) in* (1.4) *be satisfied. Then there exists a classical solution for the boundary problem* (2.17) (2.31) *in the region $q > c$ near the degenerate curve Γ.*

Remark 2. *Since the assumptions (A_2) and (A_3) in* (1.4) *are mainly used to ensure that the symbol of $\tilde{\partial}^0(q/c)$ is positive on Γ, it is obvious by* (2.30) *that, if the swirl velocity $w \neq 0$ on Γ, the condition (A_3) can be relaxed. That means the non-zero swirl component plays a positive role in ensuring the positiveness of $\tilde{\partial}^0(q/c)$.*

3 Solutions in a partial hodograph plane

In order to deal with the possible singularity caused by the hyperbolic degradation at Γ, we introduce a partial hodograph transformation to single out the feature of governing equations and then solve the problem in the partial hodograph coordinate system.

3.1 Reformulated problem in a partial hodograph plane

This subsection is devoted to reformulating the problem into a new problem in the partial hodograph plane.

3.1.1 A partial hodograph transformation

We define the coordinate transformation $(x, r) \to (t, \zeta)$ as follows

$$t = \cos\omega(x, r), \qquad \zeta = \theta(x, r). \tag{3.1}$$

It is not difficult to check by using (2.12) and (2.15) that the Jacobian of the transformation (3.1) is

$$J := \frac{\partial(t,\zeta)}{\partial(x,r)} = \frac{\tilde{\partial}^+\omega\tilde{\partial}^-\theta - \tilde{\partial}^-\omega\tilde{\partial}^+\theta}{2r^2\cos\omega} = \frac{2\sin^2\omega(\kappa+\sin^2\omega)}{r^2\cos\omega}\widetilde{J}, \tag{3.2}$$

where

$$\begin{aligned}\widetilde{J} &= X(Y-H) + Y(X+H) + \sin\theta\tilde{\partial}^0\Xi - GX\cos\omega\sin\theta \\ &\quad + \frac{G(\kappa+\sin^2\omega)}{\sin 2\omega}[(X+H)\cos\beta - (Y-H-G\cos\omega\sin\theta)\cos\alpha] - \frac{G\sin^2\theta}{2}.\end{aligned} \tag{3.3}$$

Making use of the definition of Ξ in (2.11) and (2.29), we have

$$\tilde{\partial}^0\Xi|_\Gamma = \frac{\tilde{\partial}^0\sin\omega|_\Gamma}{2(\kappa+1)} + \frac{\tilde{\partial}^0 B|_\Gamma}{4\kappa\hat{B}} = -\frac{\varphi'\cos\hat{\theta}-\sin\hat{\theta}}{\varphi'\sin\hat{\theta}+\cos\hat{\theta}}(\hat{H}-\hat{a}_0) + \frac{\hat{G}\sin\hat{\theta}}{2}.$$

Putting the above into (3.3), applying (2.28) and rearranging the result gives

$$\begin{aligned}\widetilde{J}|_\Gamma &= \left(2\hat{a}_0 - \frac{\sin\hat{\theta}(\varphi'\cos\hat{\theta}-\sin\hat{\theta})}{\cos\hat{\theta}+\varphi'\sin\hat{\theta}}\right)(\hat{H}-\hat{a}_0) + \hat{G}(\kappa+1)\cos\hat{\theta}(\hat{H}-\hat{a}_0) - \frac{\hat{G}^2(\kappa+1)\sin^2\hat{\theta}}{2} \\ &= \frac{\hat{H}-\hat{a}_0}{\cos\hat{\theta}+\varphi'\sin\hat{\theta}}\left\{\varphi\hat{\theta}' + \hat{G}(\kappa+1)\sin\hat{\theta}(\varphi'\cos\hat{\theta}-\sin\hat{\theta})\right\} - \frac{\hat{G}^2(\kappa+1)\sin^2\hat{\theta}}{2} \\ &= \frac{\hat{H}-\hat{a}_0}{\cos\hat{\theta}+\varphi'\sin\hat{\theta}}\left\{\varphi\hat{\theta}' + \frac{\hat{w}^2}{\hat{q}^2}\sin\hat{\theta}(\varphi'\cos\hat{\theta}-\sin\hat{\theta})\right\} - \frac{\hat{G}^2(\kappa+1)\sin^2\hat{\theta}}{2} \\ &\le \frac{\hat{H}-\hat{a}_0}{\cos\hat{\theta}+\varphi'\sin\hat{\theta}}\left\{\varphi\hat{\theta}' + \sin\hat{\theta}(\varphi'\cos\hat{\theta}-\sin\hat{\theta})\right\} < 0,\end{aligned} \tag{3.4}$$

by the assumptions (A_2) and (A_3) in (1.4). We combine (3.2) and (3.4) to obtain that $J < 0$ away from the degenerate curve Γ.

It follows from (2.12), (2.15) and (3.1) that

$$\begin{aligned}\tilde{\partial}^+ &= -\tfrac{2h^2(X+H)}{t}\partial_t\\&\quad -\Big\{\sqrt{1-t^2}(2Xt+\sin\zeta)-G(\kappa+1-t^2)(t\cos\zeta-\sqrt{1-t^2}\sin\zeta)\Big\}\partial_\zeta,\\ \tilde{\partial}^- &= -\tfrac{2h^2(Y-H-Gt\sin\zeta)}{t}\partial_t\\&\quad +\Big\{\sqrt{1-t^2}(2Yt+\sin\zeta)+G(\kappa+1-t^2)(t\cos\zeta+\sqrt{1-t^2}\sin\zeta)\Big\}\partial_\zeta,\\ \tilde{\partial}^0 &= -\tfrac{h^2(X+Y-Gt\sin\zeta)}{t^2}\partial_t+[\sqrt{1-t^2}(Y-X)+G(\kappa+1-t^2)\cos\zeta]\partial_\zeta,\end{aligned} \tag{3.5}$$

where $h = h(t) = \sqrt{(1-t^2)(\kappa+1-t^2)}$. Thus, in terms of the coordinates (t,ζ), system (2.17) can be transformed to a new system

$$G_t + \tfrac{\sqrt{1-t^2}(2Xt+\sin\zeta)-G(\kappa+1-t^2)(t\cos\zeta-\sqrt{1-t^2}\sin\zeta)}{2h^2(X+H)}tG_\zeta = -\tfrac{t}{2h^2(X+H)}F, \tag{3.6a}$$

$$\begin{aligned}&H_t - \tfrac{\sqrt{1-t^2}(Y-X)+G(\kappa+1-t^2)\cos\zeta}{h^2(X+Y-Gt\sin\zeta)}t^2H_\zeta\\&= -\tfrac{G\sin\zeta}{2(1-t^2)} - \tfrac{t^2}{h^2(X+Y-Gt\sin\zeta)}\Big\{-\tfrac{G^2}{2}(\kappa+1-t^2)(t\cos 2\zeta-\sqrt{1-t^2}\sin 2\zeta)\\&\quad+[(\kappa+1)\tfrac{X+Y}{t}+2\sin\zeta]H-\tfrac{\sin\zeta}{2}F+Gt\sin^2\zeta+G\sqrt{1-t^2}\sin\zeta\cos\zeta\\&\quad+G(t\sqrt{1-t^2}\cos\zeta+t^2\sin\zeta)X+\tfrac{\sin\zeta}{2}G(Y-X)\Big\},\end{aligned} \tag{3.6b}$$

$$\begin{aligned}&F_t - \tfrac{\sqrt{1-t^2}(Y-X)+G(\kappa+1-t^2)\cos\zeta}{h^2(X+Y-Gt\sin\zeta)}t^2F_\zeta\\&= -\tfrac{2G(1+\kappa G)\sin\zeta}{1-t^2} - \tfrac{2G(G\kappa+1)}{h^2(X+Y-Gt\sin\zeta)}t^2\Big\{-G(\kappa+1-t^2)(t\cos 2\zeta-\sqrt{1-t^2}\sin 2\zeta)\\&\quad+2(t\sqrt{1-t^2}\cos\zeta+t^2\sin\zeta)X+\sin\zeta(Y-X)\\&\quad+2t\sin^2\zeta+2\sqrt{1-t^2}\cos\zeta\sin\zeta-\tfrac{(\kappa+1)(X+Y)-4\kappa tG\sin\zeta}{2tG(G\kappa+1)}F\Big\},\end{aligned} \tag{3.6c}$$

$$\begin{aligned}&X_t - \tfrac{\sqrt{1-t^2}(2Yt+\sin\zeta)+G(\kappa+1-t^2)(t\cos\zeta+\sqrt{1-t^2}\sin\zeta)}{2h^2(Y-H-Gt\sin\zeta)}tX_\zeta\\&= -\tfrac{(\kappa+1-t^2)(X+H)}{h^2(Y-H-Gt\sin\zeta)}\tfrac{X+Y}{2t} - \tfrac{t}{2h^2(Y-H-Gt\sin\zeta)}\Big\{\\&\quad X\Big((2t^2-1)Y+X+2t\sin\zeta+\tfrac{t\sin\zeta-\sqrt{1-t^2}\cos\zeta}{2}\Big)\\&\quad+Y\Big(\tfrac{t\sin\zeta-\sqrt{1-t^2}\cos\zeta}{2}-2(\kappa+1-t^2)(X+H)\Big)\\&\quad+G(\kappa+1-t^2)(t\sin\zeta-\sqrt{1-t^2}\cos\zeta)X+\sin^2\zeta+\widetilde{\Phi}\Big\},\end{aligned} \tag{3.6d}$$

$$\begin{aligned}&Y_t + \tfrac{\sqrt{1-t^2}(2Xt+\sin\zeta)-G(\kappa+1-t^2)(t\cos\zeta-\sqrt{1-t^2}\sin\zeta)}{2h^2(X+H)}tY_\zeta\\&= -\tfrac{(\kappa+1-t^2)(Y-H)}{h^2(X+H)}\tfrac{X+Y}{2t} - \tfrac{t}{2h^2(X+H)}\Big\{\\&\quad Y\Big((2t^2-1)X+Y+2t\sin\zeta+\tfrac{t\sin\zeta+\sqrt{1-t^2}\cos\zeta}{2}\Big)\\&\quad+X\Big(\tfrac{t\sin\zeta+\sqrt{1-t^2}\cos\zeta}{2}-2(\kappa+1-t^2)(Y-H)\Big)\\&\quad+G(\kappa+1-t^2)(t\sin\zeta+\sqrt{1-t^2}\cos\zeta)Y+\sin^2\zeta+\widetilde{\Phi}\\&\quad-G(\kappa+1-t^2)\sin\zeta\tfrac{X+Y}{t}+G(\kappa+1-t^2)\sin\zeta Xt\Big\},\end{aligned} \tag{3.6e}$$

where

$$\begin{aligned}\widetilde{\Phi} = &- Gh^2 \sin\zeta \frac{X+Y}{t} + Gh^2 \cos\zeta \frac{Y-X}{\sqrt{1-t^2}} - 2G(\kappa+1-t^2)\cos\zeta\sqrt{1-t^2}H \\ &+ G(\kappa+1-t^2)\sin\zeta \frac{2\sqrt{1-t^2}\sin\zeta - t\cos\zeta}{\sqrt{1-t^2}} - \frac{(\kappa+1-t^2)\cos\zeta}{2\sqrt{1-t^2}}F \\ &- G^2(\kappa+1-t^2)^2 \sin\zeta \frac{t\cos\zeta - \sqrt{1-t^2}\sin\zeta}{\sqrt{1-t^2}}.\end{aligned}$$

We comment that system (3.6a)-(3.6e) has a clear singularity-regularity structure, although the expressions are quite complicated.

3.1.2 Boundary data in the (t, ζ)-plane

We now derive the boundary data for system (3.6a)-(3.6e) in the the (t, ζ) coordinates, corresponding to (2.31). Thanks to the assumption for $\hat{\theta}'$ in (1.4), we see that the smooth function $\zeta = \hat{\theta}(x)$ is strictly decreasing. Thus it exists an inverse function, denoted by $x = \hat{x}(\zeta)$ ($\zeta \in [\zeta_1, \zeta_2]$), where $\zeta_1 = \hat{\theta}(x_2)$ and $\zeta_2 = \hat{\theta}(x_1)$.

Obviously, the degenerate curve Γ: $r = \varphi(x)$ ($x \in [x_1, x_2]$) on the (x, r)-plane is transformed to a segment on $t = 0$ with $\zeta \in [\zeta_1, \zeta_2]$ on the (t, ζ)-plane. On this segment, we define the functions

$$G_0(\zeta) = \hat{G}(\hat{x}(\zeta)), \quad H_0(\zeta) = \hat{H}(\hat{x}(\zeta)), \quad F_0(\zeta) = \hat{F}(\hat{x}(\zeta)), \quad a_0(\zeta) = \hat{a}_0(\hat{x}(\zeta)).$$

Furthermore, if system (3.6a)-(3.6e) has a classical solution, then it should satisfy the following requirements

$$\begin{aligned}&\lim_{t\to 0^+} G_t(t,\zeta) = 0, \quad \lim_{t\to 0^+} H_t(t,\zeta) = -\frac{G_0(\zeta)\sin\zeta}{2}, \\ &\lim_{t\to 0^+} F_t(t,\zeta) = -2G_0(\zeta)\sin\zeta[1+\kappa G_0(\zeta)], \ \lim_{t\to 0^+} X_t(t,\zeta) = a_1(\zeta), \ \lim_{t\to 0^+} Y_t(t,\zeta) = a_1(\zeta),\end{aligned}$$

where

$$a_1(\zeta) = \frac{(\varphi'\cos\zeta - \sin\zeta)(\hat{a}_0 - \hat{H})(\hat{x}(\zeta))}{\varphi'\sin\zeta + \cos\zeta} + \frac{\sin\zeta}{2}G_0(\zeta),$$

which corresponds to the value of $(\tilde{\partial}^0 \Xi)|_\Gamma$ in the (t, ζ)-plane.

Hence, we solve system (3.6a)-(3.6e) with the boundary conditions listed as follows:

$$\begin{aligned}&(G,H,F,X,Y)(0,\zeta) = (G_0, H_0, F_0, -a_0, a_0)(\zeta), \\ &(G_t,H_t,F_t,X_t,Y_t)(0,\zeta) = (0, -\frac{\sin\zeta}{2}G_0, -2G_0(1+\kappa G_0)\sin\zeta, a_1, a_1)(\zeta)\end{aligned} \tag{3.6}$$

for $\zeta \in [\zeta_1, \zeta_2]$. It is obvious by (2.31), (2.30) and (1.4)that

$$\begin{aligned}&(G_0, H_0, F_0, a_0, a_1)(\zeta) \in C^3([\zeta_1, \zeta_2]), \\ &H_0(\zeta) - a_0(\zeta) \geq \varepsilon_0, \ \ a_1(\zeta) - \tfrac{\sin\zeta}{2}G_0(\zeta) \leq -\varepsilon_0, \ \ \forall\, \zeta \in [\zeta_1, \zeta_2]\end{aligned} \tag{3.7}$$

for some constant positive ε_0. Thus, in terms of (t, ζ)-plane, we reformulate Problem 1.1 into the following new problem.

Problem 3.1. *Under the assumption* (3.7), *we seek a local classical solution for system* (3.6a)-(3.6e) *with boundary condition* (3.6) *in the region* $t > 0$.

3.2 Existence of solutions in the partial hodograph plane

In this subsection, we apply the fixed-point method to solve the problem 3.1 in a weighted metric space.

3.2.1 The homogeneous boundary value problem

We introduce the variables $(\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y})$ to homogenize the boundary conditions (3.6)

$$\begin{gathered}\widetilde{G} = G - G_0, \quad \widetilde{H} = H - H_0 + \tfrac{\sin\zeta}{2} G_0 t, \quad \widetilde{F} = F - F_0 + 2tG_0(1 + \kappa G_0)\sin\zeta, \\ \widetilde{X} = X + a_0 - a_1 t, \quad \widetilde{Y} = Y - a_0 - a_1 t.\end{gathered} \tag{3.8}$$

Hence the boundary conditions for the variables $(\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y})$ are

$$\begin{gathered}\widetilde{G}(0,\zeta) = \widetilde{H}(0,\zeta) = \widetilde{F}(0,\zeta) = \widetilde{X}(0,\zeta) = \widetilde{Y}(0,\zeta) = 0, \\ \widetilde{G}_t(0,\zeta) = \widetilde{H}_t(0,\zeta) = \widetilde{F}_t(0,\zeta) = \widetilde{X}_t(0,\zeta) = \widetilde{Y}_t(0,\zeta) = 0,\end{gathered} \quad \forall\, \zeta \in [\zeta_1, \zeta_2]. \tag{3.9}$$

Performing a direct calculation and simplification, system (3.6a)-(3.6e) can be rewritten as

$$\begin{cases} \widetilde{G}_t + \lambda_1 \widetilde{G}_\zeta = b_1(t,\zeta), \\ \widetilde{H}_t + \lambda_2 \widetilde{H}_\zeta = b_2(t,\zeta), \\ \widetilde{F}_t + \lambda_3 \widetilde{F}_\zeta = b_3(t,\zeta), \\ \widetilde{X}_t + \lambda_4 \widetilde{X}_\zeta = \frac{\widetilde{X}+\widetilde{Y}}{2t} + b_4(t,\zeta), \\ \widetilde{Y}_t + \lambda_5 \widetilde{Y}_\zeta = \frac{\widetilde{X}+\widetilde{Y}}{2t} + b_5(t,\zeta), \end{cases} \tag{3.10}$$

where λ_i $(i = 1, \cdots, 5)$, the eigenvalues of system (3.10), are

$$\begin{aligned} \lambda_1(t,\zeta) = \lambda_5(t,\zeta) &= \frac{2t\sqrt{1-t^2}\widetilde{X} - (\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\widetilde{G} + \phi_0}{2h^2(\widetilde{X} + \widetilde{H} + \phi)} t, \\ \lambda_2(t,\zeta) = \lambda_3(t,\zeta) &= -\frac{\sqrt{1-t^2}(\widetilde{Y} - \widetilde{X}) + (\kappa + 1 - t^2)\cos\zeta\widetilde{G} + \varpi_0}{h^2[\widetilde{X} + \widetilde{Y} - t\sin\zeta\widetilde{G} + \varpi t]} t^2, \\ \lambda_4(t,\zeta) &= -\frac{2t\sqrt{1-t^2}\widetilde{Y} + (\kappa + 1 - t^2)(t\cos\zeta + \sqrt{1-t^2}\sin\zeta)\widetilde{G} + \psi_0}{2h^2(\widetilde{Y} - \widetilde{H} - t\sin\zeta\widetilde{G} + \psi)} t, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\phi = H_0 - a_0 + a_1 t - \tfrac{\sin\zeta}{2} G_0 t, \quad \psi = a_0 - H_0 + a_1 t - \tfrac{\sin\zeta}{2} G_0 t, \\ &\varpi = 2a_1 - G_0 \sin\zeta, \quad \varpi_0 = 2a_0\sqrt{1-t^2} + G_0(\kappa + 1 - t^2)\cos\zeta, \\ &\phi_0 = 2t\sqrt{1-t^2}(-a_0 + a_1 t) + \sqrt{1-t^2}\sin\zeta - G_0(\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta), \\ &\psi_0 = 2t\sqrt{1-t^2}(a_0 + a_1 t) + \sqrt{1-t^2}\sin\zeta + G_0(\kappa + 1 - t^2)(t\cos\zeta + \sqrt{1-t^2}\sin\zeta). \end{aligned}$$

The right-hand terms b_i $(i = 1, \cdots, 5)$ in system (3.10) are

$$b_1(t,\zeta) = -\frac{t}{2h^2(\widetilde{X} + \widetilde{H} + \phi)}(\widetilde{F} + b_{1a}\widetilde{X} + b_{1b}\widetilde{G} + b_{1c}),$$

$$\begin{aligned} b_2(t,\zeta) = b_{2a}\widetilde{G} - \frac{t^2}{h^2(\widetilde{X} + \widetilde{Y} - t\sin\zeta\widetilde{G} + \varpi t)} \Big\{ & b_{2b}\widetilde{X} + b_{2c}\widetilde{Y} + b_{2d}\widetilde{G} + b_{2e}\widetilde{H} + b_{2f}\widetilde{F} \\ & + b_{2g}\widetilde{G}^2 + b_{2h}\widetilde{Y}\widetilde{G} + b_{2i}\widetilde{X}\widetilde{G} + b_{2j}\frac{\widetilde{Y} + \widetilde{X}}{t}\widetilde{G} + b_{2k}\frac{\widetilde{Y} + \widetilde{X}}{t}\widetilde{H} + b_{2l}\frac{\widetilde{Y} + \widetilde{X}}{t} + b_{2m} \Big\}, \end{aligned}$$

$$\begin{aligned} b_3(t,\zeta) = & b_{3a}\widetilde{G}^2 + b_{3b}\widetilde{G} + b_{3c}t^2 - \frac{2t^2(\widetilde{G} + G_0)(\kappa\widetilde{G} + \kappa G_0 + 1)}{h^2(\widetilde{X} + \widetilde{Y} - t\sin\zeta\widetilde{G} + \varpi t)}(b_{3d}\widetilde{X} + b_{3e}\widetilde{Y} + b_{3f}\widetilde{G} + b_{3g}) \\ & + b_{3h}t^2\frac{\sqrt{1-t^2}(\widetilde{Y} - \widetilde{X}) + \widetilde{G}(\kappa + 1 - t^2)\cos\zeta + \varpi_0}{h^2(\widetilde{X} + \widetilde{Y} - \widetilde{G}t\sin\zeta + \varpi t)} \\ & + t(\widetilde{F} + b_{3i})\frac{(\kappa + 1)(\widetilde{X} + \widetilde{Y}) - 4\kappa t\widetilde{G} + b_{3j}t}{h^2(\widetilde{X} + \widetilde{Y} - \widetilde{G}t\sin\zeta + \varpi t)}, \end{aligned}$$

$$b_4(t,\zeta) = -\frac{\widetilde{X}+\widetilde{Y}+\varpi t-\widetilde{G}t\sin\zeta - t^2(\widetilde{Y}-\widetilde{H}-t\sin\zeta\widetilde{G}+\psi)}{(1-t^2)(\widetilde{Y}-\widetilde{H}-t\sin\zeta\widetilde{G}+\psi)}\left(a_1+\frac{\widetilde{X}+\widetilde{Y}}{2t}\right)$$
$$-\frac{t}{2h^2(\widetilde{Y}-\widetilde{H}-t\sin\zeta\widetilde{G}+\psi)}\Big\{b_{4a}\widetilde{X}+b_{4b}\widetilde{Y}+b_{4c}\widetilde{G}+b_{4d}\widetilde{H}+b_{4e}\widetilde{F}+\widetilde{X}^2+b_{4f}\widetilde{G}^2$$
$$+b_{4g}\widetilde{G}\widetilde{H}+b_{4h}\widetilde{X}\widetilde{Y}+b_{4i}\widetilde{X}\widetilde{G}+b_{4j}\widetilde{Y}\widetilde{G}+b_{4k}\widetilde{Y}\widetilde{H}+b_{4l}\frac{\widetilde{X}+\widetilde{Y}}{t}+b_{4m}\frac{\widetilde{X}+\widetilde{Y}}{t}\widetilde{G}+b_{4n}\Big\},$$

$$b_5(t,\zeta) = -\frac{\widetilde{X}+\widetilde{Y}-t^2(\widetilde{X}+\widetilde{H})+2a_1t-t^2\phi}{(1-t^2)(\widetilde{X}+\widetilde{H}+\phi)}\left(a_1+\frac{\widetilde{X}+\widetilde{Y}}{2t}\right)$$
$$-\frac{t}{2h^2(\widetilde{X}+\widetilde{H}+\phi)}\Big\{b_{5a}\widetilde{X}+b_{5b}\widetilde{Y}+b_{5c}\widetilde{G}+b_{5d}\widetilde{H}+b_{5e}\widetilde{F}+\widetilde{Y}^2+b_{5f}\widetilde{G}^2+b_{5g}\widetilde{G}\widetilde{H}$$
$$+b_{5h}\widetilde{X}\widetilde{Y}+b_{5i}\widetilde{X}\widetilde{G}+b_{5j}\widetilde{Y}\widetilde{G}+b_{5k}\widetilde{X}\widetilde{H}+b_{5l}\frac{\widetilde{X}+\widetilde{Y}}{t}+b_{5m}\frac{\widetilde{X}+\widetilde{Y}}{t}\widetilde{G}+b_{5n}\Big\}.$$

Here the coefficients $b_{\mu\nu}$ in b_i $(i=1,\cdots,5)$ all are C^2-smooth known functions of t and ζ, which follows from the detailed expressions of $b_{\mu\nu}$ provided in Appendix C.

Let D_δ be a region in the plane (t,ζ) defined as follows

$$D_\delta = \{(t,\zeta)|\ 0\le t\le\delta,\ \bar{\zeta}_1(t)\le\zeta\le\bar{\zeta}_2(t)\}, \tag{3.12}$$

where $\bar{\zeta}_1(t)$ and $\bar{\zeta}_2(t)$ are smooth functions satisfying $\bar{\zeta}_1(0)=\zeta_1$, $\bar{\zeta}_2(0)=\zeta_2$ and $\bar{\zeta}_1(t)<\bar{\zeta}_2(t)$ for $t\in[0,\delta]$. The functions $\bar{\zeta}_1(t)$ and $\bar{\zeta}_2(t)$ will be determined together with the solution in the iteration process. Then Problem 3.1 is equivalent to the following problem.

Problem 3.2. *Assume that* (3.7) *holds. We look for a classical solution to system* (3.10) *with homogeneous boundary conditions* (3.9) *in the region D_δ for some constant $\delta>0$.*

3.2.2 A weighted metric space

The strategy for solving Problem 3.2 is to show the existence of fixed-point to an iteration mapping in a suitable weighted metric space. Inspired by Zhang and Zheng [47], we here introduce a weighted metric space defined on the strong determinate domain of system (3.10). We first give the definition of admissible function.

Definition 3.1 (Admissible function). *We call $\mathbf{F}=(f_1(t,\zeta),f_2(t,\zeta),f_3(t,\zeta),f_4(t,\zeta),f_5(t,\zeta))^T$ $((t,\zeta)\in D_\delta)$ an admissible vector function if the following hold:*
(i) The functions f_i $(i=1,\cdots,5)$ are continuous on the region D_δ.
(ii) The functions f_i $(i=1,\cdots,5)$ satisfy the boundary value conditions in (3.9).
(iii) There holds $\sum_{i=1}^5\left\|\frac{f_i}{t^2}\right\|_\infty\le M$ for some positive constant M.

Let $\mathcal{W}_\delta^M$ be the set of all admissible vector functions. Assume that $\mathbf{F}=(f_1,f_2,f_3,f_4,f_5)^T\in\mathcal{W}_\delta^M$ and $(\tau,z)\in D_\delta$, we use $\zeta_i(t;\tau,z)$ $(i=1,\cdots,5)$ to represent the integral curves of the following equations

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\zeta_i(t;\tau,z)=\lambda_i(t,\zeta_i(t;\tau,z)),\\ \zeta_i(\tau;\tau,z)=z,\end{cases} \tag{3.13}$$

where λ_i $(i=1,\cdots,5)$ are given in (3.11) but with (f_1,f_2,f_3,f_4,f_5) replacing $(\widetilde{G},\widetilde{H},\widetilde{F},\widetilde{X},\widetilde{Y})$.

Definition 3.2 (Strong determinate domain). *The region D_δ is a strong determinate domain for system* (3.10) *if for any $\mathbf{F}\in\mathcal{W}_\delta^M$ and for any point $(\tau,z)\in D_\delta$, the curves $\zeta_i(t;\tau,z)$ $(i=1,\cdots,5)$ stay inside D_δ for all $0\le t\le\tau$ until the intersection with the line $t=0$.*

Denote $\mathcal{S}_\delta^M$ the function class of all continuously differentiable vector functions $\mathbf{F} = (f_1, f_2, f_3, f_4, f_5)^T : D_\delta \to \mathbb{R}^5$ satisfying the following properties:

$$(P_1): \ \mathbf{F}(0,\zeta) = \mathbf{F}_t(0,\zeta) = 0,$$

$$(P_2): \ \left\| \frac{\mathbf{F}(t,\zeta)}{t^2} \right\|_\infty \le M,$$

$$(P_3): \ \left\| \frac{\partial_r \mathbf{F}(t,\zeta)}{t^2} \right\|_\infty \le M,$$

$$(P_4): \ \partial_r \mathbf{F}(t,\zeta) \text{ is Lipschitz continuous with respect to r and } \left\| \frac{\partial_{\zeta\zeta}\mathbf{F}(t,\zeta)}{t^2} \right\|_\infty \le M.$$

Here $\|\cdot\|_\infty$ represents the supremum norm on the domain D_δ. It is easily seen that $\mathcal{S}_\delta^M$ is a subset of $\mathcal{W}_\delta^M$ and both $\mathcal{S}_\delta^M$ and $\mathcal{W}_\delta^M$ are subsets of $C^0(D_\delta, \mathbb{R}^5)$. Let $\mathbf{F} = (f_1, f_2, f_3, f_4, f_5)^T$ and $\mathbf{G} = (g_1, g_2, g_3, g_4, g_5)^T$ be any two elements in $\mathcal{W}_\delta^M$. We define the weighted metric

$$d(\mathbf{F},\mathbf{G}) := \sum_{i=1}^{5} \left\| \frac{f_i - g_i}{t^2} \right\|_\infty . \tag{3.14}$$

We note that $(\mathcal{W}_\delta^M, d)$ is a complete metric space, while the subset $(\mathcal{S}_\delta^M, d)$ is not closed in the space $(\mathcal{W}_\delta^M, d)$.

For Problem 3.2, we will show the following theorem

Theorem 3.1. *Suppose that conditions* (3.7) *holds and D_{δ_0} is a strong determinate domain for system* (3.10), *then the problem* (3.10) (3.9) *has a classical solution in the function class $\mathcal{S}_\delta^M$ for some positive constants $\delta \in (0, \delta_0)$ and M.*

3.2.3 The proof of Theorem 3.1

The proof of Theorem 3.1 is based on the fixed-point method. We divide the proof into four steps. Step 1 is devoted to linearizing the differential equations (3.10) to construct the integration iteration mapping. In Step 2, we derive a priori estimates for b_i and λ_i $(i = 1, \cdots, 5)$ in the function class $\mathcal{S}_\delta^M$. The existence of fixed-point for the iteration mapping is established in Step 3. Finally, we show that the limit vector function of the iteration sequence also belongs to $\mathcal{S}_\delta^M$ in Step 4.

Step 1 (The iteration mapping). Let the vector function $(\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})^T(t,\zeta) \in \mathcal{S}_\delta^M$. We consider the linearized problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}_1 t}\widetilde{G} = b_1(t,\zeta), \\ \frac{\mathrm{d}}{\mathrm{d}_2 t}\widetilde{H} = b_2(t,\zeta), \\ \frac{\mathrm{d}}{\mathrm{d}_3 t}\widetilde{F} = b_3(t,\zeta), \\ \frac{\mathrm{d}}{\mathrm{d}_4 t}\widetilde{X} = \frac{\tilde{x}+\tilde{y}}{2t} + b_4(t,\zeta), \\ \frac{\mathrm{d}}{\mathrm{d}_5 t}\widetilde{Y} = \frac{\tilde{x}+\tilde{y}}{2t} + b_5(t,\zeta), \end{cases} \tag{3.15}$$

with the boundary conditions (3.9). Here

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \partial_t + \lambda_i(t,\zeta)\partial_\zeta, \tag{3.16}$$

$\lambda_i(t,\zeta)$ and $b_i(t,\zeta)$ $(i = 1, \cdots, 5)$ are defined in (3.11) and (3.10), respectively, but with $(\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})$ replacing $(\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y})$. For any point (τ, z) in a strong determinate domain D_δ for system (3.15), we draw the characteristic curves $\zeta_i(t;\tau,z)$ $(i = 1, \cdots, 5)$ from (τ, z) up to the segment $t = 0$ and then integrate (3.15) along the characteristics to obtain

$$\begin{cases} \widetilde{G}(\tau,z) = \int_0^\tau b_1(t,\zeta_1(t;\tau,z))\,\mathrm{d}t, \\ \widetilde{H}(\tau,z) = \int_0^\tau b_2(t,\zeta_2(t;\tau,z))\,\mathrm{d}t, \\ \widetilde{F}(\tau,z) = \int_0^\tau b_3(t,\zeta_3(t;\tau,z))\,\mathrm{d}t, \\ \widetilde{X}(\tau,z) = \int_0^\tau \left(\frac{\tilde{x}+\tilde{y}}{2t} + b_4\right)(t,\zeta_4(t;\tau,z))\,\mathrm{d}t, \\ \widetilde{Y}(\tau,z) = \int_0^\tau \left(\frac{\tilde{x}+\tilde{y}}{2t} + b_5\right)(t,\zeta_5(t;\tau,z))\,\mathrm{d}t. \end{cases} \tag{3.17}$$

Thus, we get an iteration mapping $\mathcal{T}$ by (3.17)

$$\mathcal{T}\begin{pmatrix} \tilde{g} \\ \tilde{h} \\ \tilde{f} \\ \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \widetilde{G} \\ \widetilde{H} \\ \widetilde{F} \\ \widetilde{X} \\ \widetilde{Y} \end{pmatrix}. \tag{3.18}$$

Then the solving Problem 3.2 changes to show the existence of fixed point for the mapping $\mathcal{T}$ in $\mathcal{S}_\delta^M$.

Step 2 (Estimates in $\mathcal{S}_\delta^M$). We derive the estimates of b_i and λ_i ($i = 1,\cdots,5$) in (3.15) and (3.16) for later applications. Throughout the paper, we use the notation K to denote the constant depending only on ε_0, κ, r_1, r_2 and the C^3 norms of G_0, H_0, F_0, a_0, a_1, which may change from one line to the next.

According to the definitions of $h, a_1, \phi, \varpi, \psi$ and (3.7), one may choose a constant $\delta_0 < 1$ small enough such that

$$h^2 \geq \frac{1}{2}, \quad \phi \geq \frac{\varepsilon_0}{2}, \quad \varpi \leq -\varepsilon_0, \quad \psi \leq -\frac{\varepsilon_0}{2} \tag{3.19}$$

for all $(t,\zeta) \in D_{\delta_0}$. Moreover, the assumption $(\tilde{g},\tilde{h},\tilde{f},\tilde{x},\tilde{y})^T \in \mathcal{S}_\delta^M$ leads to

$$\begin{gathered} |\tilde{g}| + |\tilde{h}| + |\tilde{f}| + |\tilde{x}| + |\tilde{y}| \leq Mt^2, \quad |\tilde{g}_\zeta| + |\tilde{h}_\zeta| + |\tilde{f}_\zeta| + |\tilde{x}_\zeta| + |\tilde{y}_\zeta| \leq Mt^2, \\ |\tilde{g}_{\zeta\zeta}| + |\tilde{h}_{\zeta\zeta}| + |\tilde{f}_{\zeta\zeta}| + |\tilde{x}_{\zeta\zeta}| + |\tilde{y}_{\zeta\zeta}| \leq Mt^2. \end{gathered} \tag{3.20}$$

It is easily seen by combining with (3.19) and (3.20) that there exists a small constant $\delta_1 \leq \delta_0$ such that for $(t,\zeta) \in D_{\delta_1}$

$$\begin{gathered} |h^2[\tilde{x} + \tilde{y} - t\sin\zeta\tilde{g} + \varpi t]| \geq \tfrac{t}{2}(\varepsilon_0 - M\delta_1) \geq \tfrac{\varepsilon_0}{4}t, \\ |2h^2(\tilde{x} + \tilde{h} + \phi)| \geq \tfrac{\varepsilon_0}{2} - M\delta_1^2 \geq \tfrac{\varepsilon_0}{4}, \quad |2h^2(\tilde{y} - \tilde{h} - t\sin\zeta\tilde{g} + \psi)| \geq \tfrac{\varepsilon_0}{2} - M\delta_1^2 \geq \tfrac{\varepsilon_0}{4}. \end{gathered} \tag{3.21}$$

In addition, since $b_{\mu\nu}$ all are C^2-smooth known functions of t and ζ, then we have

$$|\partial_t b_{\mu\nu}| + |\partial_{tt} b_{\mu\nu}| + |\partial_\zeta b_{\mu\nu}| + |\partial_{\zeta\zeta} b_{\mu\nu}| \leq K. \tag{3.22}$$

We now estimate b_5, $\partial_\zeta b_5$ and $\partial_{\zeta\zeta} b_5$, the others are similar. For simplicity, we denote b_5 as

$$\begin{aligned} b_5 &= -\frac{I_1}{(1-t^2)(\tilde{x} + \tilde{h} + \phi)}\left(a_1 + \frac{\tilde{x}+\tilde{y}}{2t}\right) \\ &\quad - \frac{t}{2h^2(\tilde{x} + \tilde{h} + \phi)}\left(I_2 + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + b_{5n}\right) \end{aligned}$$

$$=: I + II, \tag{3.23}$$

where

$$\begin{aligned}I_1 &= \tilde{x} + \tilde{y} - t^2(\tilde{x} + \tilde{h}) + 2a_1 t - t^2\phi,\\ I_2 &= b_{5a}\tilde{x} + b_{5b}\tilde{y} + b_{5c}\tilde{g} + b_{5d}\tilde{h} + b_{5e}\tilde{f} + \tilde{y}^2 + b_{5f}\tilde{g}^2\\ &\quad + b_{5g}\tilde{g}\tilde{h} + b_{5h}\tilde{x}\tilde{y} + b_{5i}\tilde{x}\tilde{g} + b_{5j}\tilde{y}\tilde{g} + b_{5k}\tilde{x}\tilde{h}.\end{aligned}$$

Differentiating I and II with respect to ζ, respectively, give

$$\begin{aligned}\partial_\zeta I = &-\frac{I_1}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)}\left(a_1' + \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t}\right) - \frac{\partial_\zeta I_1}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)}\left(a_1 + \frac{\tilde{x}+\tilde{y}}{2t}\right)\\ &+ \frac{I_1(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)^2}\left(a_1 + \frac{\tilde{x}+\tilde{y}}{2t}\right),\end{aligned}$$

and

$$\begin{aligned}\partial_\zeta II = &-\frac{t}{2h^2(\tilde{x}+\tilde{h}+\phi)}\left(\partial_\zeta I_2 + \partial_\zeta(b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{t} + \partial_\zeta b_{5n}\right)\\ &+ \frac{t(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)}{2h^2(\tilde{x}+\tilde{h}+\phi)^2}\left(I_2 + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + b_{5n}\right),\end{aligned}$$

subsequently,

$$\begin{aligned}\partial_{\zeta\zeta} I = &-\frac{I_1}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)}\left(a_1'' + \frac{\tilde{x}_{\zeta\zeta} + \tilde{y}_{\zeta\zeta}}{2t}\right) - \frac{2\partial_\zeta I_1}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)}\left(a_1' + \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t}\right)\\ &+ \frac{2I_1(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)^2}\left(a_1' + \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t}\right) - \frac{\partial_{\zeta\zeta} I_1}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)}\left(a_1 + \frac{\tilde{x}+\tilde{y}}{2t}\right)\\ &+ \left\{\frac{2\partial_\zeta I_1(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)^2} + \frac{I_1(\tilde{x}_{\zeta\zeta} + \tilde{h}_{\zeta\zeta} + \phi_{\zeta\zeta})}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)^2} - \frac{2I_1(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)^2}{(1-t^2)(\tilde{x}+\tilde{h}+\phi)^3}\right\}\left(a_1 + \frac{\tilde{x}+\tilde{y}}{2t}\right),\end{aligned}$$

and

$$\begin{aligned}\partial_{\zeta\zeta} II = &-\frac{t}{2h^2(\tilde{x}+\tilde{h}+\phi)}\left(\partial_{\zeta\zeta} I_2 + \partial_{\zeta\zeta}(b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + 2\partial_\zeta(b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{t}\right.\\ &\left.+ (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}_{\zeta\zeta} + \tilde{y}_{\zeta\zeta}}{t} + \partial_{\zeta\zeta} b_{5n}\right) - \frac{t(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)^2}{h^2(\tilde{x}+\tilde{h}+\phi)^3}\left(I_2 + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + b_{5n}\right)\\ &+ \frac{t(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)}{h^2(\tilde{x}+\tilde{h}+\phi)^2}\left(\partial_\zeta I_2 + \partial_\zeta(b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{t} + \partial_\zeta b_{5n}\right)\\ &+ \frac{t(\tilde{x}_{\zeta\zeta} + \tilde{h}_{\zeta\zeta} + \phi_{\zeta\zeta})}{2h^2(\tilde{x}+\tilde{h}+\phi)^2}\left(I_2 + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x}+\tilde{y}}{t} + b_{5n}\right).\end{aligned}$$

Thanks to (3.20) and (3.22), one easy to obtain

$$|I_1| + |\partial_\zeta I_1| + |\partial_{\zeta\zeta} I_1| \le Kt(1+Mt), \quad |I_2| + |\partial_\zeta I_2| + |\partial_{\zeta\zeta} I_2| \le Kt(1+Mt)^2.$$

Making use of the above and (3.20) again, we acquire by doing simple calculations

$$\begin{aligned}|I| &\le K|I_1|(K+Mt) \le Kt(1+Mt)^2,\\ |\partial_\zeta I| &\le K|I_1|(K+Mt) + K|\partial_\zeta I_1|(K+Mt) + K|I_1|(Mt^2 + Mt^2 + K)(K+Mt)\\ &\le Kt(1+Mt)^3,\\ |\partial_{\zeta\zeta} I| &\le K|I_1|(K+Mt) + K|\partial_\zeta I_1|(K+Mt)\\ &\quad + K|I_1|(Mt^2 + Mt^2 + K)(K+Mt) + K|\partial_{\zeta\zeta} I_1|(K+Mt)\end{aligned}$$

$$
\begin{aligned}
&+ [K|\partial_\zeta I_1| + K|I_1| + K|I_1|(Mt^2 + Mt^2 + K)](Mt^2 + Mt^2 + K)(K + Mt)\\
&\le Kt(1 + Mt)^4,
\end{aligned}
$$

and

$$
\begin{aligned}
|II| &\le Kt(|I_2| + (K + KMt^2)Mt + K) \le Kt(1 + Mt)^2,\\
|\partial_\zeta II| &\le Kt[|\partial_\zeta I_2| + Mt(K + KMt^2) + (K + KMt^2)Mt + K]\\
&\quad + Kt(Mt^2 + Mt^2 + K)(|I_2| + (K + KMt^2)Mt + K)\\
&\le Kt(1 + Mt)^3,\\
|\partial_{\zeta\zeta} II| &\le Kt[|\partial_{\zeta\zeta} I_2| + (K + KMt^2)Mt + K(K + KMt^2)Mt + (K + KMt^2)Mt + K]\\
&\quad + Kt(Mt^2 + Mt^2 + K)^2[|I_2| + (K + KMt^2)Mt + K]\\
&\quad + Kt(Mt^2 + Mt^2 + K)[|\partial_\zeta I_2| + (K + KMt^2)Mt + (K + KMt^2)Mt + K]\\
&\quad + Kt(Mt^2 + Mt^2 + K)[|I_2| + (K + KMt^2)Mt + K]\\
&\le Kt(1 + Mt)^4,
\end{aligned}
$$

from which one has

$$
|b_5| \le K(1 + Mt)^2 t, \quad |\partial_\zeta b_5| \le K(1 + Mt)^3 t, \quad |\partial_{\zeta\zeta} b_5| \le K(1 + Mt)^4 t. \tag{3.24}
$$

The above estimates also hold for b_i (i=1,2,3,4).

We next estimate the eigenvalue λ_5, and λ_2, λ_4 can be discussed analogously. It follows from (3.11) that

$$
\begin{aligned}
|\lambda_5| &= \left| \frac{2t\sqrt{1 - t^2}\tilde{x} - (\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1 - t^2}\sin\zeta)\tilde{g} + \phi_0}{2h^2(\tilde{x} + \tilde{h} + \phi)} t \right|\\
&\le Kt[KMt^3 + KMt^2 + K] \le Kt(1 + Mt).
\end{aligned}
$$

Furthermore, we directly calculate

$$
\partial_\zeta \lambda_5 = \frac{2t\sqrt{1 - t^2}\tilde{x}_\zeta - (\kappa + 1 - t^2)\partial_\zeta[(t\cos\zeta - \sqrt{1 - t^2}\sin\zeta)\tilde{g}] + \phi_0'}{2h^2(\tilde{x} + \tilde{h} + \phi)} t - \frac{\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta}{\tilde{x} + \tilde{h} + \phi}\lambda_5,
$$

and

$$
\begin{aligned}
\partial_{\zeta\zeta}\lambda_5 =& \frac{2t\sqrt{1 - t^2}\tilde{x}_{\zeta\zeta} - (\kappa + 1 - t^2)\partial_{\zeta\zeta}[(t\cos\zeta - \sqrt{1 - t^2}\sin\zeta)\tilde{g}] + \phi_0''}{2h^2(\tilde{x} + \tilde{h} + \phi)} t\\
&- \frac{2t\sqrt{1 - t^2}\tilde{x}_\zeta - (\kappa + 1 - t^2)\partial_\zeta[(t\cos\zeta - \sqrt{1 - t^2}\sin\zeta)\tilde{g}] + \phi_0'}{2h^2(\tilde{x} + \tilde{h} + \phi)^2}(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi')t\\
&- \frac{\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta}{\tilde{x} + \tilde{h} + \phi}\partial_\zeta\lambda_5 - \frac{\tilde{x}_{\zeta\zeta} + \tilde{h}_{\zeta\zeta} + \phi_{\zeta\zeta}}{\tilde{x} + \tilde{h} + \phi}\lambda_5 + \frac{(\tilde{x}_\zeta + \tilde{h}_\zeta + \phi_\zeta)^2}{(\tilde{x} + \tilde{h} + \phi)^2}\lambda_5,
\end{aligned}
$$

from which one obtains

$$
\begin{aligned}
|\partial_\zeta\lambda_5| &\le Kt(KMt^3 + KMt^2 + K) + Kt(1 + Mt)(Mt^2 + Mt^2 + K) \le Kt(1 + Mt)^2,\\
|\partial_{\zeta\zeta}\lambda_5| &\le Kt(KMt^3 + KMt^2 + K) + Kt(Mt^2 + Mt^2 + K)^2\\
&\quad + K(Mt^2 + Mt^2 + K)\cdot Kt(1 + Mt)^2 + K(Mt^2 + Mt^2 + K)\cdot Kt(1 + Mt)\\
&\quad + K(Mt^2 + Mt^2 + K)^2 Kt(1 + Mt) \le Kt(1 + Mt)^3.
\end{aligned}
$$

Similar estimates also hold for the eigenvalues λ_2 and λ_4.

In summary, we have the following a priori estimates

$$
\begin{aligned}
&|b_i| \le K(1 + Mt)^2 t, \ \left|\tfrac{\partial b_i}{\partial\zeta}\right| \le K(1 + Mt)^3 t, \ \left|\tfrac{\partial^2 b_i}{\partial\zeta^2}\right| \le K(1 + Mt)^4 t,\\
&|\lambda_i| \le K(1 + Mt)t, \ \left|\tfrac{\partial\lambda_i}{\partial\zeta}\right| \le K(1 + Mt)^2 t, \ \left|\tfrac{\partial^2\lambda_i}{\partial\zeta^2}\right| \le K(1 + Mt)^3 t,
\end{aligned}
\quad i = 1, \cdots, 5. \tag{3.25}
$$

Step 3 (Properties of the mapping).

We show the following properties of the mapping $\mathcal{T}$.

Lemma 3.1. *Suppose that the assumptions in Theorem 3.1 hold. Then there exists positive constants* $\delta \in (0, \delta_1)$, M *and* $0 < \varsigma < 1$ *depending only on* $\varepsilon_0, \kappa, r_1, r_2$ *and the* C^3 *norms of* G_0, H_0, F_0, a_0, a_1 *such that*
(1) $\mathcal{T}$ *maps* $\mathcal{S}^M_\delta$ *into* $\mathcal{S}^M_\delta$;
(2) For any vector functions $\widetilde{\mathbf{F}}, \overline{\mathbf{F}}$ *in* $\mathcal{S}^M_\delta$, *there holds*

$$d\Big(\mathcal{T}(\widetilde{\mathbf{F}}), \mathcal{T}(\overline{\mathbf{F}})\Big) \le \varsigma d(\widetilde{\mathbf{F}}, \overline{\mathbf{F}}). \tag{3.26}$$

Proof. Let δ and M be two constants and $\widetilde{\mathbf{F}} = (\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})^T$ be an element in $\mathcal{S}^M_\delta$. For convenience, we denote $(F_1, F_2, F_3, F_4, F_5) = (\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y})$, $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = \mu_5 = 1$. Then the the integral system (3.17) can be rewritten as

$$F_i(\tau, z) = \int_0^\tau \Big(\mu_i \frac{\tilde{x} + \tilde{y}}{2t} + b_i\Big)(t, \zeta_i(t; \tau, z))\, \mathrm{d}t, \quad i = 1, \cdots, 5, \tag{3.27}$$

from which one has $F_i(0, z) = 0$, $(i = 1, \cdots, 5)$.

Due to (3.25), it is easily obtained that for $\tau \le \delta$

$$\begin{aligned}|F_i(\tau, z)| &\le \Big|\int_0^\tau \Big(\mu_i \Big|\frac{\tilde{x} + \tilde{y}}{2t}\Big| + |b_i|\Big)\mathrm{d}t\Big| \le \int_0^\tau \mu_i \frac{M}{2} t + K(1 + M\delta)^2 t \mathrm{d}t \\ &\le \Big(\mu_i \frac{M}{4} + K(1 + M\delta)^2\Big)\tau^2,\end{aligned}$$

which leads to

$$\sum_{i=1}^5 \frac{|F_i(\tau, z)|}{\tau^2} \le \frac{M}{2} + K(1 + M\delta)^2. \tag{3.28}$$

Furthermore, differentiating system (3.27) with respect to z yields

$$\frac{\partial F_i}{\partial z}(\tau, z) = \int_0^\tau \Big(\mu_i \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t} + \frac{\partial b_i}{\partial \zeta}\Big) \cdot \frac{\partial \zeta_i}{\partial z}\, \mathrm{d}t, \tag{3.29}$$

subsequently,

$$\frac{\partial^2 F_i}{\partial z^2}(\tau, z) = \int_0^\tau \Big\{\Big(\mu_i \frac{\tilde{x}_{\zeta\zeta} + \tilde{y}_{\zeta\zeta}}{2t} + \frac{\partial^2 b_i}{\partial \zeta^2}\Big) \cdot \Big(\frac{\partial \zeta_i}{\partial z}\Big)^2 + \Big(\mu_i \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t} + \frac{\partial b_i}{\partial \zeta}\Big) \cdot \frac{\partial^2 \zeta_i}{\partial z^2}\Big\}\mathrm{d}t, \tag{3.30}$$

where

$$\begin{aligned}\frac{\partial \zeta_i}{\partial z}(t; \tau, z) &= \exp\Big(\int_\tau^t \frac{\partial \lambda_i}{\partial \zeta}(\tau, \zeta_i(s; \tau, z))\, \mathrm{d}s\Big), \\ \frac{\partial^2 \zeta_i}{\partial z^2}(t; \tau, z) &= \int_\tau^t \frac{\partial^2 \lambda_i}{\partial \zeta^2} \cdot \frac{\partial \zeta_i}{\partial z}(s, \zeta_i(s; \tau, z))\, \mathrm{d}s \cdot \exp\Big(\int_\tau^t \frac{\partial \lambda_i}{\partial \zeta}(s, \zeta_i(s; \tau, z))\, \mathrm{d}s\Big)\end{aligned} \tag{3.31}$$

for $i = 1, \cdots, 5$. Employing the estimates for λ_i in (3.25) gives

$$\Big|\frac{\partial \zeta_i}{\partial z}\Big| \le \exp\Big(\int_0^\delta K(1 + M\delta)^2 s\, \mathrm{d}s\Big) \le e^{K(1+M\delta)^2\delta^2}, \tag{3.32}$$

and

$$\Big|\frac{\partial^2 \zeta_i}{\partial z^2}\Big| \le \int_0^\delta K(1 + M\delta)^3 t e^{K(1+M\delta)^2\delta^2}\, \mathrm{d}s \cdot e^{K(1+M\delta)^2\delta^2}$$

$$\leq K\delta^2(1+M\delta)^3 e^{K(1+M\delta)^2\delta^2}. \tag{3.33}$$

Putting (3.32) and (3.33) into (3.29) and (3.30), respectively, and using (3.25) again, we have

$$\begin{aligned}\left|\frac{\partial F_i}{\partial z}(\tau,z)\right| &\leq \int_0^\tau \left(\mu_i \frac{M}{2}t + Kt(1+M\delta)^2\right)\cdot e^{K(1+M\delta)^2\delta^2}\,dt \\ &\leq \left(\mu_i\frac{M}{4} + K(1+M\delta)^2\right)e^{K(1+M\delta)^2\delta^2}\tau^2,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2 F_i}{\partial z^2}(\tau,z)\right| \leq \int_0^\tau \Big\{&\left(\mu_i\frac{M}{2}t + K(1+M\delta)^4 t\right)\cdot e^{K(1+M\delta)^2\delta^2} \\ &+ \left(\mu_i\frac{M}{2}t + K(1+M\delta)^3 t\right)\cdot K\delta^2(1+M\delta)^3 e^{K(1+M\delta)^2\delta^2}\Big\}\,dt \\ \leq &\left(\mu_i\frac{M}{4} + K(1+M\delta)^4\right)[1+K\delta^2(1+M\delta)^3]e^{K(1+M\delta)^2\delta^2}\tau^2,\end{aligned}$$

from which one acquires

$$\sum_{i=1}^5 \frac{|\partial_z F_i(\tau,z)|}{\tau^2} \leq \left(\frac{M}{2} + K(1+M\delta)^2\right)e^{K(1+M\delta)^2\delta^2}. \tag{3.34}$$

and

$$\sum_{i=1}^5 \frac{|\partial_{zz} F_i(\tau,z)|}{\tau^2} \leq \left(\frac{M}{2} + K(1+M\delta)^4\right)[1+K\delta^2(1+M\delta)^3]e^{K(1+M\delta)^2\delta^2}. \tag{3.35}$$

For the constant $K > 1$, we select $M \geq 64K \geq 64$ and then take $\delta \leq \min\{1/M^2, \delta_1\}$, where δ_1 is given in (3.21), to obtain

$$\begin{aligned}&\left(\frac{M}{2} + K(1+M\delta)^4\right)[1+K\delta^2(1+M\delta)^3]e^{K(1+M\delta)^2\delta^2} \\ &\leq \left(\frac{M}{2}+16K\right)(1+8K\delta^2)e^{4K\delta^2} \leq \left(\frac{M}{2}+\frac{M}{4}\right)\left(1+\frac{1}{8}\right)e^{\frac{\delta}{16}} \leq \frac{27}{32}e^{\frac{1}{16}}M < M,\end{aligned} \tag{3.36}$$

which together with (3.35) finds

$$\sum_{i=1}^5 \frac{|\partial_{zz} F_i(\tau,z)|}{\tau^2} \leq M. \tag{3.37}$$

It is easy to see by (3.28) and (3.34) that

$$\sum_{i=1}^5 \frac{|F_i(\tau,z)|}{\tau^2} \leq M, \quad \sum_{i=1}^5 \frac{|\partial_z F_i(\tau,z)|}{\tau^2} \leq M \tag{3.38}$$

for choosing the constants M and δ as above. Thus we have proved the properties (P_2)-(P_4). To determine $(\widetilde{G},\widetilde{H},\widetilde{F},\widetilde{X},\widetilde{Y}) \in \mathcal{S}_\delta^M$, it suffices to check that $\widetilde{G}_\tau(0,z) = \widetilde{H}_\tau(0,z) = \widetilde{F}_\tau(0,z) = \widetilde{X}_\tau(0,z) = \widetilde{Y}_\tau(0,z) = 0$. For this end, we differentiate the integral system (3.27) with respect to τ to arrive

$$\frac{\partial F_i}{\partial \tau}(\tau,z) = \mu_i\frac{\tilde{x}+\tilde{y}}{2\tau} + b_i + \int_0^\tau \left(\mu_i\frac{\tilde{x}_\zeta+\tilde{y}_\zeta}{2t} + \frac{\partial b_i}{\partial \zeta}\right)\cdot\frac{\partial \zeta_i}{\partial \tau}\,dt, \tag{3.39}$$

where

$$\frac{\partial \zeta_i}{\partial \tau}(t;\tau,z) = -\lambda_i \frac{\partial \zeta_i}{\partial z}(t;\tau,z) \tag{3.40}$$

for $i = 1, \cdots, 5$. We recall (3.20) and (3.25) and use (3.39) to obtain directly $\widetilde{G}_\tau(0,z) = \widetilde{H}_\tau(0,z) = \widetilde{F}_\tau(0,z) = \widetilde{X}_\tau(0,z) = \widetilde{Y}_\tau(0,z) = 0$. Hence we have proved $(\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y}) \in \mathcal{S}_\delta^M$, which means that $\mathcal{T}$ do map $\mathcal{S}_\delta^M$ onto itself.

We next establish (3.26) for some positive constant $\varsigma < 1$. Let $\widetilde{\mathbf{F}} = (\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})^T$ and $\overline{\mathbf{F}} = (\bar{g}, \bar{h}, \bar{f}, \bar{x}, \bar{y})^T$ be two elements in $\mathcal{S}_\delta^M$. Denote $(F_1, F_2, F_3, F_4, F_5) = (\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y}) = \mathcal{T}(\widetilde{\mathbf{F}})$, $(\overline{F_1}, \overline{F_2}, \overline{F_3}, \overline{F_4}, \overline{F_5}) = (\overline{G}, \overline{H}, \overline{F}, \overline{X}, \overline{Y}) = \mathcal{T}(\overline{\mathbf{F}})$. In view of (3.15), one has

$$\frac{\mathrm{d}}{\mathrm{d}_i t} F_i = \mu_i \frac{\tilde{x} + \tilde{y}}{2t} + b_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, \zeta), \quad \frac{\mathrm{d}}{\mathrm{d}_{\bar{i}} t} \overline{F_i} = \mu_i \frac{\bar{x} + \bar{y}}{2t} + b_i(\bar{g}, \bar{h}, \bar{x}, \bar{y}, t, \zeta), \quad (i = 1, \cdots, 5),$$

which combined with (3.16) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_i t}(F_i - \overline{F_i}) =& \frac{\mathrm{d}}{\mathrm{d}_i t} F_i + \lambda_i(t,\zeta)\overline{F}_{i\zeta} - \frac{\mathrm{d}}{\mathrm{d}_{\bar{i}} t}\overline{F_i} - \lambda_{\bar{i}}(t,\zeta)\overline{F}_{i\zeta} \\
=& \mu_i \frac{(\tilde{x} - \bar{x}) + (\tilde{y} - \bar{y})}{2t} + [b_i(\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})(t,\zeta) - b_i(\bar{g}, \bar{h}, \bar{f}, \bar{x}, \bar{y})(t,\zeta)] \\
& + [\lambda_i(t,\zeta) - \lambda_{\bar{i}}(t,\zeta)]\overline{F}_{i\zeta} \\
:=& III + IV + V.
\end{aligned} \tag{3.41}$$

For the term III, we easy to see that

$$|III| \le \mu_i \frac{|\tilde{x} - \bar{x}| + |\tilde{y} - \bar{y}|}{2t} \le \frac{\mu_i t}{2} d(\widetilde{\mathbf{F}}, \overline{\mathbf{F}}). \tag{3.42}$$

For the terms IV and V, we derive the case $i = 5$, the other cases can be treated in a similar way. Recalling the expression of b_5 in (3.23) suggests

$$\begin{aligned}
IV =& \frac{\bar{I}_1 - I_1}{(1 - t^2)(\bar{x} + \bar{h} + \phi)}\left(a_1 + \frac{\bar{x} + \bar{y}}{2t}\right) + \frac{I_1}{(1 - t^2)(\bar{x} + \bar{h} + \phi)} \frac{(\bar{x} - \tilde{x}) + (\bar{y} - \tilde{y})}{2t} \\
& + I_1\left(a_1 + \frac{\tilde{x} + \tilde{y}}{2t}\right) \frac{(\tilde{x} - \bar{x}) + (\tilde{h} - \bar{h})}{(1 - t^2)(\bar{x} + \bar{h} + \phi)(\tilde{x} + \tilde{h} + \phi)} \\
& + \frac{t}{2h^2(\bar{x} + \bar{h} + \phi)}\left\{(\bar{I}_2 - I_2) + (b_{5l} + b_{5m}\tilde{g})\frac{(\bar{x} - \tilde{x}) + (\bar{y} - \tilde{y})}{t} + b_{5m}(\bar{g} - \tilde{g})\frac{\bar{x} + \bar{y}}{t}\right\} \\
& + \frac{t}{2h^2}\left[I_2 + (b_{5l} + b_{5m}\tilde{g})\frac{\tilde{x} + \tilde{y}}{t} + b_{5n}\right] \frac{(\tilde{x} - \bar{x}) + (\tilde{h} - \bar{h})}{(\bar{x} + \bar{h} + \phi)(\tilde{x} + \tilde{h} + \phi)},
\end{aligned}$$

where I_1 and I_2 are given in (3.23), and $\bar{I}_1$ and $\bar{I}_2$ are, respectively, I_1 and I_2 but with $(\bar{g}, \bar{h}, \bar{f}, \bar{x}, \bar{y})$ replacing $(\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})$. According to the detailed expressions of I_1 and I_2 give

$$\begin{aligned}
\bar{I}_1 - I_1 =& (1 - t^2)(\bar{x} - \tilde{x}) + (\bar{y} - \tilde{y}) - t^2(\bar{h} - \tilde{h}), \\
\bar{I}_2 - I_2 =& [b_{5a} + b_{5h}\bar{y} + b_{5i}\bar{g} + b_{5k}\bar{h}](\bar{x} - \tilde{x}) + [b_{5b} + (\bar{y} + \tilde{y}) + b_{5h}\tilde{x} + b_{5j}\bar{g}](\bar{y} - \tilde{y}) \\
& + [b_{5c} + b_{5f}(\bar{g} + \tilde{g}) + b_{5g}\bar{h} + b_{5i}\tilde{x} + b_{5j}\tilde{y}](\bar{g} - \tilde{g}) \\
& + [b_{5d} + b_{5g}\tilde{g} + b_{5k}\tilde{x}](\bar{h} - \tilde{h}) + b_{5e}(\bar{f} - \tilde{f}),
\end{aligned}$$

from which we conclude that

$$|\bar{I}_1 - I_1| \le t^2 d(\widetilde{\mathbf{F}}, \overline{\mathbf{F}}), \quad |\bar{I}_2 - I_2| \le K(1 + M\delta)t^2 d(\widetilde{\mathbf{F}}, \overline{\mathbf{F}}).$$

Therefore, we obtain

$$|IV| \le K(1 + M\delta)|\bar{I}_1 - I_1| + K|I_1| t d(\widetilde{\mathbf{F}}, \overline{\mathbf{F}}) + |I_1| K(1 + M\delta)t^2 d(\widetilde{\mathbf{F}}, \overline{\mathbf{F}})$$

$$
\begin{aligned}
&+ Kt[|\bar{I}_2 - I_2| + K(1+M\delta)td(\widetilde{\mathbf{F}},\overline{\mathbf{F}}) + KM\delta t^2 d(\widetilde{\mathbf{F}},\overline{\mathbf{F}})] \\
&+ Kt[|I_2| + K(1+M\delta)^2 + K]t^2 d(\widetilde{\mathbf{F}},\overline{\mathbf{F}}) \\
\leq &K(1+M\delta)^3 t^2 d(\widetilde{\mathbf{F}},\overline{\mathbf{F}}).
\end{aligned} \tag{3.43}
$$

Moreover, we estimate V by the expression of λ_5 in (3.11) to get

$$
\begin{aligned}
|V| \leq &\Bigg| \frac{2t\sqrt{1-t^2}(\tilde{x}-\bar{x}) - (\kappa+1-t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)(\tilde{g}-\bar{g})}{2h^2(\tilde{x}+\tilde{h}+\phi)} t\overline{F_{5\zeta}} \\
&+ \frac{2t\sqrt{1-t^2}\bar{x} - (\kappa+1-t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\bar{g} + \phi_0}{2h^2(\tilde{x}+\tilde{h}+\phi)(\bar{x}+\bar{h}+\phi)} [(\bar{x}-\tilde{x}) + (\bar{h}-\tilde{h})]t\overline{F_{5\zeta}} \Bigg| \\
\leq &K(1+M\delta)^3 e^{K(1+M\delta)^2\delta^2} t^2 d(\widetilde{\mathbf{F}},\overline{\mathbf{F}}).
\end{aligned} \tag{3.44}
$$

Combining with (3.41)-(3.44) renders

$$
\begin{aligned}
|F_i - \overline{F_i}| &\leq \int_0^\tau (|III| + |IV| + |V|)\,\mathrm{d}t \\
&\leq \int_0^\tau \left\{ \frac{\mu_i}{2} t + [KM(1+M\delta)^2 t^2 + KM(1+M\delta)t^2] e^{K(1+M\delta)^2}\delta^2 \right\} d(\widetilde{\mathbf{F}},\overline{\mathbf{F}})\,\mathrm{d}t \\
&\leq \tau^2 \left\{ \frac{\mu_i}{4} + K\delta(1+M\delta)^3 e^{K(1+M\delta)^2\delta^2} \right\} d(\widetilde{\mathbf{F}},\overline{\mathbf{F}}),
\end{aligned}
$$

from which one has

$$
d\Big(\mathcal{T}(\widetilde{\mathbf{F}}), \mathcal{T}(\overline{\mathbf{F}})\Big) = \sum_{i=1}^{5} \frac{|F_i - \overline{F_i}|}{\tau^2} \leq \left\{ \frac{1}{2} + K\delta(1+M\delta)^3 e^{K(1+M\delta)^2\delta^2} \right\} d(\widetilde{\mathbf{F}},\overline{\mathbf{F}}) \leq \frac{3}{4} d(\widetilde{\mathbf{F}},\overline{\mathbf{F}}),
$$

by choosing δ as in (3.36), which completes the proof of (3.26). Thus $\mathcal{T}$ is a contraction under the metric d. □

Step 4 (Properties of the limit function). In this step, we affirm that the limit vector function obtained in Step 3 is also in $\mathcal{S}_\delta^M$, which follows directly from the Arzela-Ascoli theorem and the following lemma.

Lemma 3.2. *Let the assumptions in Theorem 3.1 hold. Suppose that $\widetilde{\mathbf{F}} = (\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})^T \in \mathcal{S}_\delta^M$ and denote $(F_1, F_2, F_3, F_4, F_5) = (\widetilde{G}, \widetilde{H}, \widetilde{F}, \widetilde{X}, \widetilde{Y}) = \mathcal{T}(\widetilde{\mathbf{F}})$. Then $\partial_t F_i(t,\zeta)$ and $\partial_\zeta F_i(t,\zeta)$ $(i = 1, \cdots, 5)$ are uniformly Lipschitz continuous on D_δ.*

Proof. Since $(\tilde{g}, \tilde{h}, \tilde{f}, \tilde{x}, \tilde{y})^T \in \mathcal{S}_\delta^M$, then by Lemma 3.1 we see that $(F_1, F_2, F_3, F_4, F_5)$ is also in $\mathcal{S}_\delta^M$. We show the lemma in three steps.

Firstly, we assert that the inequalities $|\partial_t F_i| \leq Mt$ $(i = 1, \cdots, 5)$ hold. Indeed, by recalling (3.39) and using the estimates of (3.25), (3.32) and (3.40), one can easily obtain

$$
\begin{aligned}
\left|\frac{\partial F_i}{\partial \tau}(\tau, z)\right| &\leq \left|\mu_i \frac{\tilde{x}+\tilde{y}}{2\tau}\right| + |b_i| + \int_0^\tau \left(\mu_i \left|\frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t}\right| + \left|\frac{\partial b_i}{\partial \zeta}\right| \right) \cdot \left|\frac{\partial \zeta_i}{\partial \tau}\right| \mathrm{d}t \\
&\leq \frac{1}{2}M\tau + K(1+M\delta)^2\tau + K\tau^2(1+M\delta)^4 e^{K(1+M\delta)^2\delta^2} \leq M\tau.
\end{aligned} \tag{3.45}
$$

Secondly, we verified the inequalities $|\partial_{t\zeta} F_i| \leq Mt$ $(i = 1, \cdots, 5)$. Differentiating (3.39) with respect to z gives

$$
\begin{aligned}
\frac{\partial^2 F_i}{\partial z \partial \tau}(\tau, z) = &\left(\mu_i \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2\tau} + \frac{\partial b_i}{\partial \zeta} \right) \frac{\partial \zeta_i}{\partial z} \\
&+ \int_0^\tau \left\{ \left(\mu_i \frac{\tilde{x}_{\zeta\zeta} + \tilde{y}_{\zeta\zeta}}{2t} + \frac{\partial^2 b_i}{\partial \zeta^2} \right) \frac{\partial \zeta_i}{\partial z} \frac{\partial \zeta_i}{\partial \tau} + \left(\mu_i \frac{\tilde{x}_\zeta + \tilde{y}_\zeta}{2t} + \frac{\partial b_i}{\partial \zeta} \right) \frac{\partial^2 \zeta_i}{\partial \tau \partial z} \right\} \mathrm{d}t,
\end{aligned}
$$

where

$$\frac{\partial^2\zeta_i}{\partial\tau\partial z}(t;x,r)=\exp\left(\int_\tau^t\frac{\partial\lambda_i}{\partial\zeta}\,\mathrm{d}s\right)\cdot\left\{\int_\tau^t\frac{\partial^2\lambda_i}{\partial\zeta^2}\cdot\frac{\partial\zeta_i}{\partial\tau}\,\mathrm{d}s-\frac{\partial\lambda_i}{\partial\zeta}\right\}.$$

Noting the fact

$$\begin{aligned}\left|\frac{\partial^2\zeta_i}{\partial\tau\partial z}\right|&\le e^{K(1+M\delta)^2\delta^2}\cdot\left\{\int_0^\delta K(1+M\delta)^4e^{K(1+M\delta)^2\delta^2}s^2\,\mathrm{d}s+K(1+M\delta)^2t\right\}\\&\le K\delta(1+M\delta)^4e^{K(1+M\delta)^2\delta^2},\end{aligned}\tag{3.46}$$

we find that

$$\begin{aligned}\left|\frac{\partial^2F_i}{\partial z\partial\tau}\right|&\le\left(\frac{\mu_i}{2}M\tau+K(1+M\delta)^3\tau+K(1+M\delta)^3\tau\right)e^{K(1+M\delta)^2\delta^2}\\&\quad+\int_0^\tau\Bigg\{\left(\frac{\mu_i}{2}Mt+K(1+M\delta)^4t\right)e^{K(1+M\delta)^2\delta^2}K(1+M\delta)t\\&\qquad+\left(\frac{\mu_i}{2}Mt+K(1+M\delta)^3t\right)K\delta(1+M\delta)^4e^{K(1+M\delta)^2\delta^2}\Bigg\}\\&\le\tau\left(\frac{1}{2}M+K(1+M\delta)^4\right)\left(1+K\delta^2(1+M\delta)^4\right)e^{K(1+M\delta)^2\delta^2}\le M\tau.\end{aligned}\tag{3.47}$$

Finally, we claim that the inequalities $|\partial_{tt}F_i|\le 4M$ $(i=1,\cdots,5)$ hold. We differentiate (3.39) with respect to τ to know

$$\begin{aligned}\frac{\partial^2F_i}{\partial\tau^2}(\tau,z)&=\mu_i\frac{\tilde{x}_\tau+\tilde{y}_\tau}{\tau}-\mu_i\frac{\tilde{x}+\tilde{y}}{\tau^2}+2\frac{\partial b_i}{\partial\tau}\\&\quad+\int_0^\tau\left\{\left(\mu_i\frac{\tilde{x}_\zeta+\tilde{y}_\zeta}{2t}+\frac{\partial b_i}{\partial\zeta}\right)\frac{\partial^2\zeta_i}{\partial\tau^2}+\left(\mu_i\frac{\tilde{x}_{\zeta\zeta}+\tilde{y}_{\zeta\zeta}}{2t}+\frac{\partial^2b_i}{\partial\zeta^2}\right)\left(\frac{\partial\zeta_i}{\partial\tau}\right)^2\right\}\mathrm{d}t,\end{aligned}$$

which together with the estimate $|\partial_tF_i|\le Mt$ gets

$$\begin{aligned}\left|\frac{\partial^2F_i}{\partial\tau^2}\right|&\le 3M+2\left|\frac{\partial b_i}{\partial\tau}\right|+\int_0^\delta\Bigg\{t\left(\frac{1}{2}M+K(1+M\delta)^3\right)\left|\frac{\partial^2\zeta_i}{\partial\tau^2}\right|\\&\qquad+\left(\frac{1}{2}M+K(1+M\delta)^4\right)K(1+M\delta)^2e^{K(1+M\delta)^2\delta^2}t^3\Bigg\}\,\mathrm{d}t.\end{aligned}\tag{3.48}$$

For the term $\frac{\partial b_i}{\partial\tau}$, we have

$$\left|\frac{\partial b_i}{\partial\tau}\right|\le K\delta(1+M\delta)^3e^{K(1+M\delta)^2\delta^2}.\tag{3.49}$$

The derivation of (3.49) is directly from the expression of b_i and the previous estimates, we omit it here. To deal with $|\frac{\partial^2\zeta_i}{\partial\tau^2}|$, we recall (3.40) and employ (3.32) and (3.46) to obtain

$$\begin{aligned}\left|\frac{\partial^2\zeta_i}{\partial\tau^2}\right|&=\left|-\frac{\partial\lambda_i}{\partial\tau}\cdot\frac{\partial\zeta_i}{\partial z}-\lambda_i\frac{\partial^2\zeta_i}{\partial\tau\partial z}\right|\\&\le\left(\left|\frac{\partial\lambda_i}{\partial\tau}\right|+K\delta^2(1+M\delta)^5\right)e^{K(1+M\delta)^2\delta^2}.\end{aligned}\tag{3.50}$$

Differentiating λ_i with respect to t and simplifying the result, we can get the estimate of $\partial_t\lambda_i$

$$\left|\frac{\partial\lambda_i}{\partial t}\right|\le K(1+M\delta)^3.$$

which along with (3.50) gives

$$\left|\frac{\partial^2 \zeta_i}{\partial \tau^2}\right| \le K(1+M\delta)^4 e^{K(1+M\delta)^2\delta^2}. \tag{3.51}$$

We combine (3.48), (3.49) and (3.51) to conclude

$$\begin{aligned}\left|\frac{\partial^2 F_i}{\partial \tau^2}\right| &\le 3M + K\delta(1+M\delta)^3 e^{K(1+M\delta)^2\delta^2} \\ &\quad + K\delta^2[M + K(1+M\delta)^4](1+M\delta)^4 e^{K(1+M\delta)^2\delta^2} \le 4M.\end{aligned} \tag{3.52}$$

Summing up (3.45), (3.47) and (3.52) and applying Lemma 3.1, we finish the proof of Lemma 3.2, and then complete the proof of Theorem 3.1. □

4 Solutions in the physical plane

In this section, we convert the solution in the (t, ζ)-plane to that in the original (x, r)-plane. Note that Problem 3.2 and Problem 3.1 are equivalent, then from Theorem 3.1, we know that system (3.6a)-(3.6e) with the boundary data (3.6) admits a local classical solution $(G, H, F, X, Y)(t, \zeta)$.

In view of the coordinate transformation (3.1), one has

$$\frac{\partial x}{\partial t} = \frac{\theta_r}{J}, \quad \frac{\partial r}{\partial t} = -\frac{\theta_x}{J}, \quad \frac{\partial x}{\partial \zeta} = \frac{\sin\omega\omega_r}{J}, \quad \frac{\partial r}{\partial \zeta} = -\frac{\sin\omega\omega_x}{J}, \tag{4.1}$$

where J is defined in (3.2). On the other hand, according to (2.8), (2.12) and (2.15) deriving

$$\begin{aligned} r\theta_x &= (t\sin\zeta - \sqrt{1-t^2}\cos\zeta)X + (t\sin\zeta + \sqrt{1-t^2}\cos\zeta)Y + (\kappa+1-t^2)G + \sin^2\zeta, \\ r\theta_r &= -(t\cos\zeta + \sqrt{1-t^2}\sin\zeta)X - (t\cos\zeta - \sqrt{1-t^2}\sin\zeta)Y - \cos\zeta\sin\zeta, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} r\omega_x &= -\tfrac{k+1-t^2}{t^2}\Big\{(t\sin\zeta - \sqrt{1-t^2}\cos\zeta)X - (t\sin\zeta + \sqrt{1-t^2}\cos\zeta)Y \\ &\qquad +2t\sin\zeta H + t\sin\zeta G(t\sin\zeta + \sqrt{1-t^2}\cos\zeta)\Big\}, \\ r\omega_r &= \tfrac{k+1-t^2}{t^2}\Big\{(t\cos\zeta + \sqrt{1-t^2}\sin\zeta)X - (t\cos\zeta - \sqrt{1-t^2}\sin\zeta)Y \\ &\qquad +2t\cos\zeta H + t\sin\zeta G(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\Big\}. \end{aligned} \tag{4.3}$$

Hence, with the help of (4.1) and (4.2), we can obtain the functions $x = x(t, \zeta)$ and $r = r(t, \zeta)$ by solving the following differential equations

$$\begin{cases} \frac{\partial r}{\partial t}(t,\zeta) = -\frac{r[(t\sin\zeta - \sqrt{1-t^2}\cos\zeta)X + (t\sin\zeta + \sqrt{1-t^2}\cos\zeta)Y + (\kappa+1-t^2)G + \sin^2\zeta]}{2h^2\Psi(t,\zeta)}t, \\ r(0,\zeta) = \varphi(\hat{\theta}^{-1}(\zeta)), \end{cases} \tag{4.4}$$

and

$$\begin{cases} \frac{\partial x}{\partial t}(t,\zeta) = -\frac{r[(t\cos\zeta + \sqrt{1-t^2}\sin\zeta)X + (t\cos\zeta - \sqrt{1-t^2}\sin\zeta)Y + \cos\zeta\sin\zeta]}{2h^2\Psi(t,\zeta)}t, \\ x(0,\zeta) = \hat{\theta}^{-1}(\zeta), \end{cases} \tag{4.5}$$

where

$$\begin{aligned} \Psi(t,\zeta) &= X(Y - H - t\sin\zeta G) + Y(X+H) + \frac{X+Y}{2t}\sin\zeta \\ &\quad + \frac{(\kappa+1-t^2)}{2t\sqrt{1-t^2}}G\Big\{t\cos\zeta(X - Y + 2H + Gt\sin\zeta) + \sqrt{1-t^2}\sin\zeta(X + Y - Gt\sin\zeta)\Big\}, \end{aligned}$$

and $\hat{\theta}^{-1}$ represents the inverse of $\hat{\theta}$. The well-posedness of equations (4.4) and (4.5) comes from the fact by (3.4)

$$\Psi(t,\zeta) \le -\varepsilon < 0, \quad \forall (t,\zeta) \in D_\delta$$

for some small constant $\varepsilon > 0$. Furthermore, we construct the function $w(t,\zeta)$ by solving the linear problem

$$\begin{cases} w_t - \frac{\sqrt{1-t^2}(Y-X)+G(\kappa+1-t^2)\cos\zeta}{h^2(X+Y-Gt\sin\zeta)} t^2 w_\zeta = -\frac{t^2}{h^2(X+Y-Gt\sin\zeta)} w \sin\zeta, \\ w(0,\zeta) = \hat{w}(\hat{\theta}^{-1}(\zeta)), \end{cases}$$

which is derived from (2.9), (3.5) and (2.19). In the same way, we solve the liner problems

$$\begin{cases} B_t - \frac{\sqrt{1-t^2}(Y-X)+G(\kappa+1-t^2)\cos\zeta}{h^2(X+Y-Gt\sin\zeta)} t^2 B_\zeta = -\frac{t^2}{h^2(X+Y-Gt\sin\zeta)} w^2 \sin\zeta, \\ B(0,\zeta) = \hat{B}(\hat{\theta}^{-1}(\zeta)), \end{cases}$$

and

$$\begin{cases} S_t - \frac{\sqrt{1-t^2}(Y-X)+G(\kappa+1-t^2)\cos\zeta}{h^2(X+Y-Gt\sin\zeta)} t^2 S_\zeta = 0, \\ S(0,\zeta) = \hat{S}(\hat{\theta}^{-1}(\zeta)) \end{cases}$$

to obtain the functions of $B(t,\zeta)$ and $S(t,\zeta)$, respectively. In addition, we find by using (4.1)-(4.3) that the Jacobian of the map $(t,r) \mapsto (x,r)$ is

$$j := \frac{\partial(x,r)}{\partial(t,\zeta)} = \frac{r^2 t}{2h^2\Psi},$$

which is strictly less than zero when $t \in (0,\delta]$. This means that the map $(t,\zeta) \mapsto (x,r)$ is an one-to-one mapping for $t \in (0,\delta]$. Therefore we obtain the functions $t = t(x,r)$ and $\zeta = \zeta(x,r)$, and then define by (3.1)

$$\begin{aligned} &\theta = \zeta(x,r),\ \omega = \arccos t(x,r),\ B = B(t(x,r),\zeta(x,r)),\ S = S(t(x,r),\zeta(x,r)), \\ &w = w(t(x,r),\zeta(x,r)),\ \alpha = \zeta(x,r) + \arccos t(x,r),\ \beta = \zeta(x,r) - \arccos t(x,r). \end{aligned} \tag{4.6}$$

To complete the proof of Theorem 2.1, it suffices to determine that the functions defined in (4.6) satisfy system (2.9). For the equation of S, we use (4.1)-(4.3) to compute

$$\begin{aligned} \tilde{\partial}^0 S &= r[\cos\zeta(S_t t_x + S_\zeta \zeta_x) + \sin\zeta(S_t t_r + S_\zeta \zeta_r)] \\ &= \frac{r}{j}[(\cos\zeta r_\zeta - \sin\zeta x_\zeta)S_t - (\cos\zeta r_t - \sin\zeta x_t)S_\zeta] \\ &= -\sqrt{1-t^2}[\cos\zeta(r\omega_x) + \sin\zeta(r\omega_r)]S_t + [\cos\zeta(r\theta_x) + \sin\zeta(r\theta_r)]S_\zeta \\ &= -\frac{h^2(X+Y-Gt\sin\zeta)}{t^2}\left\{S_t - \frac{G(\kappa+1-t^2)\cos\zeta + \sqrt{1-t^2}(Y-X)}{h^2(X+Y-Gt\sin\zeta)} t^2 S_\zeta\right\} = 0. \end{aligned}$$

The checking of the equations for B and w are analogous. We now examine the fourth equation of (2.9). By a direct calculation, one has

$$\begin{aligned} \tilde{\partial}^+\theta + \frac{\cos^2\omega}{\kappa+\sin^2\omega}\tilde{\partial}^+\omega &= r\left\{(\cos\alpha\zeta_x + \sin\alpha\zeta_r) - \frac{t^2\sqrt{1-t^2}}{h^2}(\cos\alpha t_x + \sin\alpha t_r)\right\} \\ &= \frac{r}{j}\left\{-\cos\alpha r_t + \sin\alpha x_t - \frac{t^2\sqrt{1-t^2}}{h^2}(\cos\alpha r_\zeta - \sin\alpha x_\zeta)\right\} \\ &= [\cos\alpha(r\theta_x) + \sin\alpha(r\theta_r)] + \frac{t^2(1-t^2)}{h^2}[\cos\alpha(r\omega_x) + \sin\alpha(r\omega_r)]. \end{aligned} \tag{4.7}$$

Moreover, making use of (4.2)-(4.3) and (4.6) arrives at

$$\begin{aligned} \cos\alpha(r\theta_x) + \sin\alpha(r\theta_r) &= G(\kappa+1-t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta) - t\sin\zeta, \\ \cos\alpha(r\omega_x) + \sin\alpha(r\omega_r) &= \frac{2h^2}{t\sqrt{1-t^2}}(X+H). \end{aligned}$$

We put the above into (4.7) to find

$$\begin{aligned}&\tilde{\partial}^{+}\theta+\frac{\cos^{2}\omega}{\kappa+\sin^{2}\omega}\tilde{\partial}^{+}\omega\\ =&G(\kappa+1-t^{2})(t\cos\zeta-\sqrt{1-t^{2}}\sin\zeta)-t\sin\zeta+\frac{t^{2}(1-t^{2})}{h^{2}}\cdot\frac{2h^{2}}{t\sqrt{1-t^{2}}}(X+H)\\ =&2t\sqrt{1-t^{2}}(X+H)+G(\kappa+1-t^{2})(t\cos\zeta-\sqrt{1-t^{2}}\sin\zeta)-t\sin\zeta.\end{aligned}$$

which is the desired equation by the definition of H. The last equation of (2.9) can be checked similarly. In summary, the functions $(S,B,w,\theta,\omega)(x,r)$ constructed in (4.6) satisfy the equations (2.9).

Finally, we can define the functions $(\rho,u,v,w,p)(x,r)$ by the definitions of $(S,B,w,\theta,\omega)(x,r)$

$$c=\frac{\sin\omega(x,r)\sqrt{2kB(x,r)}}{\sqrt{\kappa+\sin^{2}\omega(x,r)}},\quad u=c(x,r)\frac{\cos\theta(x,r)}{\sin\omega(x,r)},\quad v=c(x,r)\frac{\sin\theta(x,r)}{\sin\omega(x,r)},\quad w=w(x,r)$$
$$\rho=\left(\frac{2kB(x,r)\sin^{2}\omega(x,r)}{\gamma[\kappa+\sin^{2}\omega(x,r)]S(x,r)}\right)^{\frac{1}{\gamma-1}},\quad p=S(x,r)\left(\frac{2kB(x,r)\sin^{2}\omega(x,r)}{\gamma[\kappa+\sin^{2}\omega(x,r)]S(x,r)}\right)^{\frac{\gamma}{\gamma-1}}.$$

One can check that the functions $(\rho,u,v,w,p)(x,r)$ defined as above satisfy system (2.1) subject to the boundary condition $(\hat{\rho},\hat{u},\hat{v},\hat{w},\hat{p})(x)$ on the sonic curve Γ. Therefore the proof of Theorem 1.1 is complete.

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Appendices

A The derivation of system (2.9)

We here just provide the derivation of the fourth equation of (2.9). The others can be easily or in parallel obtained. We rewrite the fourth equation of (2.9) into

$$\gamma pv(u_{x}+\Lambda_{+}u_{r})-\gamma pu(v_{x}+\Lambda_{+}v_{r})-c\sqrt{q^{2}-c^{2}}(p_{x}+\Lambda_{+}p_{r})=-\frac{\gamma p(w^{2}+v^{2}-uv\Lambda_{+})}{r}.$$

According to (2.6) and (2.7), we acquire

$$c\frac{\sin\theta}{\sin\omega}\tilde{\partial}^{+}\left(c\frac{\cos\theta}{\sin\omega}\right)-c\frac{\cos\theta}{\sin\omega}\tilde{\partial}^{+}\left(c\frac{\sin\theta}{\sin\omega}\right)-c^{2}\frac{\cos\omega}{\sin\omega}\frac{\tilde{\partial}^{+}p}{\gamma p}=-[w^{2}+v(v-u\Lambda_{+})]\cos\alpha, \tag{A.1}$$

On the other hand, one uses the definition of pseudo-Bernoulli function to get

$$B=\frac{q^{2}}{2}+\frac{c^{2}}{\gamma-1}=\left(\frac{1}{2\sin^{2}\omega}+\frac{1}{\gamma-1}\right)c^{2}=\frac{\gamma(\kappa+\sin^{2}\omega)}{2\kappa\sin^{2}\omega}\cdot\frac{p}{\rho},$$

which combined with the entropy function $S=p\rho^{-\gamma}$ gives

$$\ln p=\frac{\gamma}{2\kappa}\left(\ln B-\frac{1}{\gamma}\ln S-\ln\frac{\gamma(\kappa+\sin^{2}\omega)}{2\kappa\sin^{2}\omega}\right),$$

from which one has

$$\frac{1}{\gamma p}\tilde{\partial}^{+}p=\frac{1}{2\kappa}\tilde{\partial}^{+}(\ln B-\frac{1}{\gamma}\ln S)+\frac{\cot\omega}{\kappa+\sin^{2}\omega}\tilde{\partial}^{+}\omega.$$

We insert the above into (A.1) to arrive

$$c^{2}\frac{\sin\theta}{\sin\omega}\cdot\frac{-\sin\theta\sin\omega\tilde{\partial}^{+}\theta-\cos\theta\cos\omega\tilde{\partial}^{+}\omega}{\sin^{2}\omega}-c^{2}\frac{\cos\theta}{\sin\omega}\cdot\frac{\cos\theta\sin\omega\tilde{\partial}^{+}\theta-\sin\theta\cos\omega\tilde{\partial}^{+}\omega}{\sin^{2}\omega}$$

$$-c^2\cot\omega\cdot\left[\frac{1}{2\kappa}\tilde{\partial}^+\left(\ln B-\frac{1}{\gamma}\ln S\right)+\frac{\cot\omega}{\kappa+\sin^2\omega}\tilde{\partial}^+\omega\right]=-w^2\cos\alpha+\frac{c^2\sin\theta}{\sin\omega},$$

that is,

$$\tilde{\partial}^+\theta+\frac{\cos^2\omega}{\kappa+\sin^2\omega}\tilde{\partial}^+\omega-\frac{\sin(2\omega)}{4\kappa}\tilde{\partial}^+\left(\frac{1}{\gamma}\ln S-\ln B\right)=G(\kappa+\sin^2\omega)\cos\alpha-\sin\omega\sin\theta.$$

B The derivation of system (2.17)

We here only present the derivations of the third and fourth equations in (2.17), the other three equations are analogous.

To derive the third equation, we first calculate some relations

$$\begin{aligned}
\tilde{\partial}^0G&=-2G(1+\kappa G)\sin\theta\\
\tilde{\partial}^+\theta&=-\sin(2\omega)X+(\kappa+\sin^2\omega)\cos\alpha G-\sin\theta\sin\omega\\
\tilde{\partial}^+\tilde{\partial}^0G&=-2(1+2\kappa G)\sin\theta F-2G(1+\kappa G)\cos\theta\tilde{\partial}^+\theta\\
\cos\omega\tilde{\partial}^0\alpha-\tilde{\partial}^+\theta&=\sin\omega\cos\omega(X+Y)+\frac{\sin\omega(\kappa+\sin^2\omega)(X+Y)}{\cos\omega}+\sin\omega\sin\theta\\
\cos\omega\tilde{\partial}^+\theta-\tilde{\partial}^0\alpha&=-\frac{(\kappa+\sin^2\omega)\cos\alpha\sin^2\omega}{\cos\omega}G-2\sin\omega\cos^2\omega X+\sin\omega(X-Y)\\
&\quad-\frac{(\kappa+\sin^2\omega)\sin\omega}{\cos^2\omega}(X+Y)-\sin\theta\sin\omega\cos\omega.
\end{aligned}$$

Moreover, we use the commutator relation between $\tilde{\partial}^0$ and $\tilde{\partial}^+$ in (2.16) and recall the definition of F in (2.13) to obtain

$$\tilde{\partial}^0F-\tilde{\partial}^+\tilde{\partial}^0G=\frac{\cos\omega\tilde{\partial}^0\alpha-\tilde{\partial}^+\theta}{\sin\omega}F+\frac{\cos\omega\tilde{\partial}^+\theta-\tilde{\partial}^0\alpha}{\sin\omega}\tilde{\partial}^0G+\sin\theta F-\sin\alpha\tilde{\partial}^0G.$$

Thus we have

$$\begin{aligned}
\tilde{\partial}^0F&=-2G(1+\kappa G)\cos\theta[-\sin(2\omega)X+(\kappa+\sin^2\omega)\cos\alpha G-\sin\theta\sin\omega]\\
&\quad-2(1+2\kappa G)\sin\theta F+\left(\frac{(\kappa+1)}{\cos\omega}(X+Y)+\sin\theta\right)F+\sin\theta F\\
&\quad+\left\{\frac{-G(\kappa+\sin^2\omega)\cos\alpha\sin\omega}{\cos\omega}-\cos 2\omega X-\frac{(\kappa+\sin^2\omega)}{\cos^2\omega}(X+Y)\right.\\
&\quad\left.-Y-\sin\theta\cos\omega-\sin\alpha\right\}[-2G(1+\kappa G)\sin\theta]\\
&=\frac{2G(1+\kappa G)(\kappa+\sin^2\omega)}{\cos\omega}\left(\sin\theta\frac{X+Y}{\cos\omega}-G\cos^2\alpha\right)+\frac{(\kappa+1)(X+Y)-4\kappa G\sin\theta\cos\omega}{\cos\omega}F\\
&\quad+2G(1+\kappa G)[\sin(2\omega+\theta)X+\sin\theta Y+2\sin\theta\sin\alpha],
\end{aligned}$$

which is the third equation of (2.17).

For deriving the equation for X, we apply the commutator relation between $\tilde{\partial}^-$ and $\tilde{\partial}^+$ in (2.16) to find that

$$\begin{aligned}
&\tilde{\partial}^-\tilde{\partial}^+\theta-\tilde{\partial}^+\tilde{\partial}^-\theta\\
&=\frac{\cos(2\omega)\tilde{\partial}^-\alpha-\tilde{\partial}^+\beta}{\sin(2\omega)}\tilde{\partial}^+\theta+\frac{\cos(2\omega)\tilde{\partial}^+\beta-\tilde{\partial}^-\alpha}{\sin(2\omega)}\tilde{\partial}^-\theta+\sin\beta\tilde{\partial}^+\theta-\sin\alpha\tilde{\partial}^-\theta,
\end{aligned}\tag{B.1}$$

and

$$\tilde{\partial}^-X-\tilde{\partial}^+Y=\frac{\cos(2\omega)\tilde{\partial}^-\alpha-\tilde{\partial}^+\beta}{\sin(2\omega)}X+\frac{\cos(2\omega)\tilde{\partial}^+\beta-\tilde{\partial}^-\alpha}{\sin(2\omega)}Y+\sin\beta X-\sin\alpha Y,\tag{B.2}$$

On the other hand, differentiating (2.12) along the directions $\tilde{\partial}^-$ and $\tilde{\partial}^+$, respectively, yield

$$\begin{aligned}
\tilde{\partial}^-\tilde{\partial}^+\theta =& -2\cos(2\omega)X\tilde{\partial}^-\omega - \sin(2\omega)\tilde{\partial}^-X - \cos\theta\sin\omega\tilde{\partial}^-\theta - \sin\theta\cos\omega\tilde{\partial}^-\omega \\
&+ \frac{w(\kappa+\sin^2\omega)\cos\alpha}{\kappa B}\tilde{\partial}^-w + 2\sin\omega\cos\omega\cos\alpha G\tilde{\partial}^-\omega - (\kappa+\sin^2\omega)\sin\alpha G\tilde{\partial}^-\alpha \\
&- (\kappa+\sin^2\omega)\cos\alpha G\tilde{\partial}^-\ln B, \\
\tilde{\partial}^+\tilde{\partial}^-\theta =& 2\cos(2\omega)Y\tilde{\partial}^+\omega + \sin(2\omega)\tilde{\partial}^+Y + \cos\theta\sin\omega\tilde{\partial}^+\theta + \sin\theta\cos\omega\tilde{\partial}^+\omega \\
&+ \frac{w(\kappa+\sin^2\omega)\cos\beta}{\kappa B}\tilde{\partial}^+w + 2\sin\omega\cos\omega\cos\beta G\tilde{\partial}^+\omega - (\kappa+\sin^2\omega)\sin\beta G\tilde{\partial}^+\beta \\
&- (\kappa+\sin^2\omega)\cos\beta G\tilde{\partial}^+\ln B.
\end{aligned}$$

It follows by putting the above into (B.1) that

$$\begin{aligned}
&\sin(2\omega)(\tilde{\partial}^-X + \tilde{\partial}^+Y) \\
&= -2\cos(2\omega)(X\tilde{\partial}^-\omega + Y\tilde{\partial}^-\omega) - \cos\theta\sin\omega(\tilde{\partial}^-\theta + \tilde{\partial}^+\theta) - \sin\theta\cos\omega(\tilde{\partial}^-\omega + \tilde{\partial}^+\omega) \\
&\quad + \frac{w(\kappa+\sin^2\omega)}{\kappa B}(\cos\alpha\tilde{\partial}^-w - \cos\beta\tilde{\partial}^+w) + 2\sin\omega\cos\omega G(\cos\alpha\tilde{\partial}^-\omega - \cos\beta\tilde{\partial}^+\omega) \\
&\quad - (\kappa+\sin^2\omega)G(\sin\alpha\tilde{\partial}^-\alpha - \sin\beta\tilde{\partial}^+\beta) - (\kappa+\sin^2\omega)G(\cos\alpha\tilde{\partial}^-\ln B - \cos\beta\tilde{\partial}^+\ln B) \\
&\quad - \frac{\cos(2\omega)\tilde{\partial}^-\alpha - \tilde{\partial}^+\beta}{\sin(2\omega)}\tilde{\partial}^+\theta - \frac{\cos(2\omega)\tilde{\partial}^+\beta - \tilde{\partial}^-\alpha}{\sin(2\omega)}\tilde{\partial}^-\theta - \sin\beta\tilde{\partial}^+\theta + \sin\alpha\tilde{\partial}^-\theta,
\end{aligned}$$

which together with (B.2) and (2.12) arrives at

$$\begin{aligned}
\tilde{\partial}^-X =& \frac{\cos(2\omega)\tilde{\partial}^-\alpha - \cos(2\omega)\tilde{\partial}^-\omega - \tilde{\partial}^+\beta}{\sin(2\omega)}X - \frac{\cos(2\omega)\tilde{\partial}^+\omega}{\sin(2\omega)}Y + \frac{\sin^2\theta}{2} \\
&+ \frac{\sin\beta}{2}X - \frac{\sin\alpha}{2}Y + \left[\frac{\sin\theta\cos\omega}{2} - \frac{\sin\theta(\kappa+\sin^2\omega)}{2\cos\omega}\right](X+Y) \\
&- \frac{\sin\theta\sin\omega(1+\cos(2\omega))(\tilde{\partial}^+\beta - \tilde{\partial}^-\alpha)}{2\sin^2(2\omega)} + \frac{w(\kappa+\sin^2\omega)}{2\kappa B\sin(2\omega)}(\cos\alpha\tilde{\partial}^-w - \cos\beta\tilde{\partial}^+w) \\
&- \frac{G\cos\alpha(\kappa+\sin^2\omega)(\cos(2\omega)\tilde{\partial}^-\alpha - \tilde{\partial}^+\beta)}{2\sin^2(2\omega)} - \frac{G\cos\beta(\kappa+\sin^2\omega)(\cos(2\omega)\tilde{\partial}^+\beta - \tilde{\partial}^-\alpha)}{2\sin^2(2\omega)} \\
&+ \frac{G}{2}(\cos\alpha\tilde{\partial}^-\omega - \cos\beta\tilde{\partial}^+\omega) - \frac{(\kappa+\sin^2\omega)G}{2\sin(2\omega)}(\sin\alpha\tilde{\partial}^-\alpha - \sin\beta\tilde{\partial}^+\beta) \\
&- \frac{(\kappa+\sin^2\omega)G}{2\sin(2\omega)}(\cos\alpha\tilde{\partial}^-\ln B - \cos\beta\tilde{\partial}^+\ln B) + G\sin^2\theta(\kappa+\sin^2\omega).
\end{aligned} \tag{B.3}$$

Furthermore, we compute directly

$$\begin{aligned}
\tilde{\partial}^+\beta - \tilde{\partial}^-\alpha &= -\frac{2(\kappa+1)\sin\omega}{\cos\omega}(X+Y) - 2\sin\theta\sin\omega, \\
\frac{\cos(2\omega)\tilde{\partial}^-\alpha - \cos(2\omega)\tilde{\partial}^-\omega - \tilde{\partial}^+\beta}{\sin(2\omega)} &= X + \cos(2\omega)Y + G(\kappa+\sin^2\omega)\sin\beta \\
&\quad + \frac{(\kappa+\sin^2\omega)}{\cos^2\omega}(X+H) + \sin\theta\cos\omega,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{G\cos\alpha(\kappa+\sin^2\omega)(\cos(2\omega)\tilde{\partial}^-\alpha - \tilde{\partial}^+\beta)}{2\sin^2(2\omega)} + \frac{G\cos\beta(\kappa+\sin^2\omega)(\cos(2\omega)\tilde{\partial}^+\beta - \tilde{\partial}^-\alpha)}{2\sin^2(2\omega)} \\
&= -\frac{(\kappa+\sin^2\omega)G}{2\sin(2\omega)}(\sin\alpha\tilde{\partial}^-\alpha - \sin\beta\tilde{\partial}^+\beta).
\end{aligned}$$

Hence we insert the above into (B.3) to see that

$$\tilde{\partial}^-X = (\kappa+\sin^2\omega)(X+H)\frac{X+Y}{\cos^2\omega} + X[\cos(2\omega)Y + X + 2\sin\theta\cos\omega + \frac{\sin\beta}{2}]$$

$$
\begin{aligned}
&+ Y[\frac{\sin\beta}{2} - 2(\kappa + \sin^2\omega)(X + H)] + \sin^2\theta + G\sin\beta(\kappa + \sin^2\omega)X \\
&+ G\sin^2\theta(\kappa + \sin^2\omega) + \frac{G}{2}[\cos\theta\cos\omega(\tilde{\partial}^-\omega - \tilde{\partial}^+\omega) - \sin\omega\sin\theta(\tilde{\partial}^-\omega + \tilde{\partial}^+\omega)] \\
&+ \frac{w(\kappa + \sin^2\omega)}{2\kappa B\sin(2\omega)}[\cos\theta\cos\omega(\tilde{\partial}^- w - \tilde{\partial}^+ w) - \sin\omega\sin\theta(\tilde{\partial}^- w + \tilde{\partial}^+ w)] \\
&- \frac{(\kappa + \sin^2\omega)G}{2}[\cos\theta\frac{\tilde{\partial}^-\ln B - \tilde{\partial}^+\ln B}{2\sin\omega} - \sin\theta\tilde{\partial}^0(\ln B)].
\end{aligned} \tag{B.4}
$$

In addition, we have the following relations

$$
\begin{aligned}
\tilde{\partial}^-\ln B - \tilde{\partial}^+\ln B &= 8\kappa H - 2\tilde{\partial}^+(\frac{1}{\gamma}\ln S) + 4\kappa G\cos\omega\sin\theta, \\
w\tilde{\partial}^- w - w\tilde{\partial}^+ w &= -2w^2\sin\theta\cos\omega - 2\kappa B[F + G(\tilde{\partial}^+(\frac{1}{\gamma}\ln S) - 4\kappa H)], \\
\tilde{\partial}^-\omega - \tilde{\partial}^+\omega &= \frac{2\sin\omega(\kappa + \sin^2\omega)}{\cos\omega}(Y - X - 2H - G\cos\omega\sin\theta), \\
\tilde{\partial}^-\omega + \tilde{\partial}^+\omega &= \frac{2\sin\omega(\kappa + \sin^2\omega)}{\cos\omega}(Y + X - G\cos\omega\sin\theta).
\end{aligned}
$$

Therefore, we finally obtain after doing a simplification

$$
\begin{aligned}
\tilde{\partial}^- X =& (\kappa + \sin^2\omega)(X + H)\frac{X + Y}{\cos^2\omega} + X[\cos(2\omega)Y + X + 2\sin\theta\cos\omega + \frac{\sin\beta}{2}] \\
&+ Y[\frac{\sin\beta}{2} - 2(\kappa + \sin^2\omega)(X + H)] + \sin^2\theta + G\sin\beta(\kappa + \sin^2\omega)X \\
&- G\sin^2\omega\sin\theta(\kappa + \sin^2\omega)\frac{X + Y}{\cos\omega} + G(Y - X)\sin\omega\cos\theta(\kappa + \sin^2\omega) \\
&- 2GH\sin\omega\cos\theta(\kappa + \sin^2\omega) - G^2\frac{(\kappa + \sin^2\omega)^2\sin\theta\cos\alpha}{\sin\omega} \\
&+ G\frac{\sin\theta(\kappa + \sin^2\omega)}{\sin\omega}(2\sin\theta\sin\omega - \cos\theta\cos\omega) - F\frac{(\kappa + \sin^2\omega)\cos\theta}{2\sin\omega},
\end{aligned}
$$

which is the equation for X in (2.17).

C The expressions of $b_{\mu\nu}$ in system (3.10)

We here list the detailed expressions of $b_{\mu\nu}$ in system (3.10), from which we are easy to see these terms are all C^2-smooth known functions by the boundary conditions (3.7).

The functions of $b_{1\nu}$ $(\nu = a, b, c)$ are

$$
\begin{aligned}
&b_{1a} = 2G_0' t\sqrt{1 - t^2}, \; b_{1b} = -G_0'(\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1 - t^2}\sin\zeta), \\
&b_{1c} = F_0 - 2tG_0(1 + \kappa G_0)\sin\zeta + G_0'\phi_0.
\end{aligned}
$$

The functions of $b_{2\nu}$ $(\nu = a \cdots l)$ are listed as follows

$$
\begin{aligned}
b_{2a} &= -\frac{\sin\zeta}{2(1 - t^2)}, \\
b_{2b} &= \sqrt{1 - t^2}[H_0' - \frac{t}{2}(G_0\sin\zeta)'] + G_0(t\sqrt{1 - t^2}\cos\zeta + t^2\sin\zeta) - \frac{G_0\sin\zeta}{2} + \frac{h^2 G_0\sin\zeta}{2(1 - t^2)}, \\
b_{2c} &= -\sqrt{1 - t^2}[H_0' - \frac{t}{2}(G_0\sin\zeta)'] + \frac{G_0\sin\zeta}{2} + \frac{h^2 G_0\sin\zeta}{2(1 - t^2)}, \\
b_{2d} &= -(\kappa + 1 - t^2)\cos\zeta[H_0' - \frac{t}{2}(G_0\sin\zeta)'] - G_0(\kappa + 1 - t^2)(t\cos 2\zeta - \sqrt{1 - t^2}\sin 2\zeta) \\
&\quad + t\sin^2\zeta + a_0\sin\zeta + \sqrt{1 - t^2}\sin\zeta\cos\zeta + (t\sqrt{1 - t^2}\cos\zeta + t^2\sin\zeta)(a_1 t - a_0)
\end{aligned}
$$

$$- \frac{th^2 G_0 \sin^2\zeta}{2(1-t^2)},$$

$$b_{2e} = 2a_1(\kappa+1) + 2\sin\zeta,\ b_{2f} = -\frac{\sin\zeta}{2},\ b_{2g} = -\frac{1}{2}(\kappa+1-t^2)(t\cos 2\zeta - \sqrt{1-t^2}\sin 2\zeta),$$

$$b_{2h} = \frac{\sin\zeta}{2}, b_{2i} = t\sqrt{1-t^2}\cos\zeta + t^2\sin\zeta - \frac{\sin\zeta}{2},$$

$$b_{2j} = 0,\ b_{2k} = (\kappa+1),\ b_{2l} = (\kappa+1)(H_0 - \frac{t\sin\zeta}{2}G_0),$$

$$\begin{aligned} b_{2m} = {} & -\varpi_0[H_0' - \frac{t}{2}(G_0\sin\zeta)'] + a_0G_0\sin\zeta \\ & - \frac{1}{2}G_0^2(\kappa+1-t^2)(t\cos 2\zeta - \sqrt{1-t^2}\sin 2\zeta) + [2a_1(\kappa+1) + 2\sin\zeta](H_0 - \frac{t\sin\zeta}{2}G_0) \\ & - \frac{\sin\zeta}{2}[F_0 - 2t\sin\zeta G_0(\kappa G_0 + 1)] + G_0 t\sin^2\zeta + G_0\sqrt{1-t^2}\sin\zeta\cos\zeta \\ & + G_0(a_1t - a_0)(t\sqrt{1-t^2}\cos\zeta + t^2\sin\zeta) + \frac{th^2G_0\sin\zeta\varpi}{2(1-t^2)}. \end{aligned}$$

The expressions of $b_{3\nu}$ $(\nu = a\cdots j)$ are

$$b_{3a} = -\frac{2\kappa\sin\zeta}{1-t^2},\ b_{3b} = -\frac{2\sin\zeta + 4\kappa G_0\sin\zeta}{1-t^2},\ b_{3c} = -\frac{2G_0(1+\kappa G_0)\sin\zeta}{1-t^2},$$

$$b_{3d} = 2t\sqrt{1-t^2}cos\zeta + (2t^2-1)\sin\zeta,\ b_{3e} = \sin\zeta,$$

$$b_{3f} = -(\kappa+1-t^2)(t\cos 2\zeta - \sqrt{1-t^2}\sin 2\zeta),$$

$$\begin{aligned} b_{3g} = {} & 2\sqrt{1-t^2}\cos\zeta\sin\zeta - G_0(\kappa+1-t^2)(t\cos 2\zeta - \sqrt{1-t^2}\sin 2\zeta) \\ & + 2(-a_0 + a_1t)(t\sqrt{1-t^2}\cos\zeta + t^2\sin\zeta) + 2t\sin^2\zeta + 2a_0\sin\zeta, \end{aligned}$$

$$b_{3h} = F_0' - 2t(G_0(1+\kappa G_0)\sin\zeta)',\ b_{3i} = F_0 - 2G_0(1+\kappa G_0)t\sin\zeta,$$

$$b_{3j} = 2(\kappa+1)a_1 - 4\kappa G_0\sin\zeta.$$

The functions of $b_{4\nu}$ $(\nu = a\cdots n)$ are

$$\begin{aligned} b_{4a} = {} & G_0(\kappa+1-t^2)(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) + (2t^2-1)(a_0 + a_1t) - 2(a_0 - a_1t) \\ & + 2t\sin\zeta + \frac{1}{2}(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) - 2(a_0 + a_1t)(\kappa+1-t^2) - \frac{G_0h^2\cos\zeta}{\sqrt{1-t^2}}, \end{aligned}$$

$$\begin{aligned} b_{4b} = {} & -2t\sqrt{1-t^2}(a_1't - a_0') + (2t^2-1)(-a_0 + a_1t) \\ & + \frac{1}{2}(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) - 2(\kappa+1-t^2)\phi + \frac{G_0h^2\cos\zeta}{\sqrt{1-t^2}}, \end{aligned}$$

$$\begin{aligned} b_{4c} = {} & (-a_0 + a_1t)(\kappa+1-t^2)(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) \\ & - (2H_0 - tG_0\sin\zeta)\sqrt{1-t^2}(\kappa+1-t^2)\cos\zeta \\ & - (a_1't - a_0')(\kappa+1-t^2)(t\cos\zeta + \sqrt{1-t^2}\sin\zeta) - 2a_1h^2\sin\zeta + \frac{2a_0h^2\cos\zeta}{\sqrt{1-t^2}} \\ & + \frac{(\kappa+1-t^2)\sin\zeta}{\sqrt{1-t^2}}\{2\sqrt{1-t^2}\sin\zeta - t\cos\zeta - 2G_0(\kappa+1-t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\}, \end{aligned}$$

$$b_{4d} = -2(a_0 + a_1t)(\kappa+1-t^2) - 2G_0\cos\zeta\sqrt{1-t^2}(\kappa+1-t^2), b_{4e} = -\frac{(\kappa+1-t^2)\cos\zeta}{2\sqrt{1-t^2}},$$

$$b_{4f} = -\frac{(\kappa+1-t^2)^2(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\sin\zeta}{\sqrt{1-t^2}},$$

$$b_{4g} = -2\cos\zeta\sqrt{1-t^2}(\kappa+1-t^2),\ b_{4h} = 2t^2 - 1 - 2(\kappa+1-t^2),$$

$$b_{4i} = (\kappa+1-t^2)(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) - \frac{h^2\cos\zeta}{\sqrt{1-t^2}}, b_{4j} = \frac{h^2\cos\zeta}{\sqrt{1-t^2}}, b_{4k} = -2(\kappa+1-t^2),$$

$$b_{4l} = -h^2G_0\sin\zeta, \quad b_{4m} = -h^2\sin\zeta,$$

$$\begin{aligned}
b_{4n} &= G_0(-a_0 + a_1 t)(\kappa + 1 - t^2)(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) - (a_1' t - a_0')\psi_0 \\
&\quad + (-a_0 + a_1 t)[(2t^2 - 1)(a_0 + a_1 t) + (-a_0 + a_1 t) + 2t\sin\zeta + \frac{1}{2}(t\sin\zeta - \sqrt{1-t^2}\cos\zeta)] \\
&\quad + (a_0 + a_1 t)[\frac{1}{2}(t\sin\zeta - \sqrt{1-t^2}\cos\zeta) - 2(\kappa + 1 - t^2)\phi] + \sin^2\zeta \\
&\quad - 2a_1 h^2 G_0 \sin\zeta + \frac{2a_0 h^2 G_0\cos\zeta}{\sqrt{1-t^2}} - \sqrt{1-t^2}G_0\cos\zeta(\kappa + 1 - t^2)(2H_0 - tG_0\sin\zeta) \\
&\quad + \frac{(\kappa + 1 - t^2)G_0\sin\zeta}{\sqrt{1-t^2}}\{2\sqrt{1-t^2}\sin\zeta - G_0(\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta) \\
&\quad - t\cos\zeta\} - \frac{\cos\zeta(\kappa + 1 - t^2)[F_0 - 2t\sin\zeta G_0(1 + \kappa G_0)]}{2\sqrt{1-t^2}}.
\end{aligned}$$

The expressions of $b_{5\nu}$ $(\nu = a \cdots n)$ are

$$\begin{aligned}
b_{5a} &= 2t\sqrt{1-t^2}(a_1' t + a_0') + (2t^2 - 1)(a_0 + a_1 t) + \frac{1}{2}(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) \\
&\quad - 2(\kappa + 1 - t^2)(\psi + tG_0\sin\zeta) + G_0 t\sin\zeta(\kappa + 1 - t^2) - \frac{G_0 h^2\cos\zeta}{\sqrt{1-t^2}}, \\
b_{5b} &= G_0(\kappa + 1 - t^2)(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) + (a_0 - a_1 t)(2\kappa + 3 - 4t^2) \\
&\quad + 2(a_0 + a_1 t) + 2t\sin\zeta + \frac{1}{2}(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) + \frac{G_0 h^2\cos\zeta}{\sqrt{1-t^2}}, \\
b_{5c} &= (a_0 + a_1 t)(\kappa + 1 - t^2)(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) + \frac{2a_0 h^2\cos\zeta}{\sqrt{1-t^2}} \\
&\quad - (a_1' t + a_0')(\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta) - 2a_1\sin\zeta(\kappa + 1 - t^2) - 2a_1 h^2\sin\zeta \\
&\quad + t\sin\zeta(-a_0 + a_1 t)(\kappa + 1 - t^2) - (2H_0 - G_0 t\sin\zeta)(\kappa + 1 - t^2)\sqrt{1-t^2}\cos\zeta \\
&\quad + \frac{(\kappa + 1 - t^2)\sin\zeta}{\sqrt{1-t^2}}\{2\sqrt{1-t^2}\sin\zeta - t\cos\zeta - 2G_0(\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\}, \\
b_{5d} &= 2(-a_0 + a_1 t)(\kappa + 1 - t^2) - 2G_0\cos\zeta\sqrt{1-t^2}(\kappa + 1 - t^2), \quad b_{5e} = -\frac{(\kappa + 1 - t^2)\cos\zeta}{2\sqrt{1-t^2}}, \\
b_{5f} &= -\frac{(\kappa + 1 - t^2)^2(t\cos\zeta - \sqrt{1-t^2}\sin\zeta)\sin\zeta}{\sqrt{1-t^2}}, \\
b_{5g} &= -2\cos\zeta\sqrt{1-t^2}(\kappa + 1 - t^2), \quad b_{5h} = 2t\sqrt{1-t^2} - 2(\kappa + 1 - t^2), \\
b_{5i} &= t\sin\zeta(\kappa + 1 - t^2) - \frac{h^2\cos\zeta}{\sqrt{1-t^2}}, \quad b_{5j} = (\kappa + 1 - t^2)(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) + \frac{h^2\cos\zeta}{\sqrt{1-t^2}}, \\
b_{5k} &= 2(\kappa + 1 - t^2), \quad b_{5l} = (t^2 - 2)G_0(\kappa + 1 - t^2)\sin\zeta, \quad b_{5m} = (t^2 - 2)(\kappa + 1 - t^2)\sin\zeta, \\
b_{5n} &= G_0(a_0 + a_1 t)(\kappa + 1 - t^2)(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) + (a_1' t + a_0')\phi_0 \\
&\quad + (a_0 + a_1 t)[(2t^2 - 1)(-a_0 + a_1 t) + (a_0 + a_1 t) + 2t\sin\zeta + \frac{1}{2}(t\sin\zeta + \sqrt{1-t^2}\cos\zeta)] \\
&\quad + (-a_0 + a_1 t)[\frac{1}{2}(t\sin\zeta + \sqrt{1-t^2}\cos\zeta) - 2(\kappa + 1 - t^2)(\psi + G_0 t\sin\zeta)] + \sin^2\zeta \\
&\quad - 2a_1 G_0(\kappa + 1 - t^2)\sin\zeta + G_0 t\sin\zeta(-a_0 + a_1 t)(\kappa + 1 - t^2) - 2a_1 h^2 G_0\sin\zeta \\
&\quad + \frac{2a_0 h^2 G_0\cos\zeta}{\sqrt{1-t^2}} - 2\sqrt{1-t^2}G_0\cos\zeta(\kappa + 1 - t^2)(H_0 - \frac{G_0 t\sin\zeta}{2}) \\
&\quad + \frac{(\kappa + 1 - t^2)G_0\sin\zeta}{\sqrt{1-t^2}}\{2\sqrt{1-t^2}\sin\zeta - G_0(\kappa + 1 - t^2)(t\cos\zeta - \sqrt{1-t^2}\sin\zeta) \\
&\quad - t\cos\zeta\} - \frac{\cos\zeta(\kappa + 1 - t^2)[F_0 - 2t\sin\zeta G_0(1 + \kappa G_0)]}{2\sqrt{1-t^2}}.
\end{aligned}$$

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