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Research paper

Percolation in multilayer complex networks with connectivity and interdependency topological structures

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A B S T R A C T

The concept of a multilayer network describes a typical class of networks that have multiple types of links, that represent the different natures of interactions among nodes. In this work, we investigate the cascading dynamics in double-layer networks with two topological layers composed of connectivity links and interdependency links. The failure of a node can disable the nodes that disconnect the viable nodes, but it can also cause some amount of damage to its interdependency neighbours. We find that the characteristics of the percolation transition can be categorized into three types: first-order, second-order and double phase transition, which depend on the interdependency strength among nodes and the density of interdependency links of the system. We develop a theoretical framework to predict the percolation transition points and the switching point of percolation types. We have also validated our model in a double-layer empirical network composed by an internet and a power grid, and found that the results reproduced by our model is in concordance with the observations of cascading failures occurred in the critical infrastructure systems. Our work not only gives a possible qualitative explanation for the unexpected large-scale damages or disruptive avalanches in real-world infrastructure systems, but it also provides enlightening significance for how the double-layer network can be designed to have a satisfying resilience level.

1. Introduction

It has been widely believed that the study of plenty of complicated phenomena in real word complex systems needs the concept of multilayer network [1], because of the existence of various kinds of links that represent different interaction mechanisms among nodes. Real instances of multilayer network are numerous, such as social [2], technological [3–5], and biological systems [6,7]. A multilayer network is sometimes tantamount to a network of networks where the failure of one node might destroy a node in another layer, if the nodes belonging to different layers are regarded as being interdependent on each other. The robustness of multilayer networks or the network of networks has become one hot research field in recent years [8–16]. In multilayer networks, failure can propagate from one node to other nodes and cause large-scale failures in an avalanching manner throughout different layers by the different interaction mechanisms that is represented. Therefore, a multilayer networked system [8,17,18] always disintegrates discontinuously, so it is more fragile than a single-layer

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network [19–22]. The theoretical works on cascading failures in multilayer networks or the network of networks have revealed that some topological properties, such as inter-layer degree correlations [23,24], intra-layer degree correlations [25], link overlap [26,27], spatial constraints [28–31], clustering [32,33], and degree distribution [34,35], significantly influence the robustness properties of multilayer networks. According to the interaction types of nodes across network layers, some works have also explored the properties of percolation transitions of multilayer networks by extended percolation models, such as *k*-core [36], weak [37], and redundant [38] percolations.

In most existing models of multilayer networks, the multiple sets of links are all connection type and are homogeneous in essence, where a node is viable if it is connected to the giant component by every set of links. When such a multilayer network suffers from malicious attack or random failures, a node will fail if it is isolated from the giant component by any type of links. Otherwise, the node will maintain its function perfectly even if a mass of neighbours are removed. However, it has been widely accepted that the components in a complex system are always interconnected and interdependent on each other simultaneously, and the removal of a node will destroy or impair the functions of nodes in its neighbourhood even if they can connect to the giant component [39–46]. For instance, in a cyberphysical system consisting of a power grid and a communication network, the failure of a power station may have a significant influence to its neighbours in the power grid due to the overload that is caused by the redistribution of electrical current or voltage [47–50]. There exists a substantial interdependency between a pair of neighbouring nodes in the power grid. However, two neighbouring nodes in the communication layer are not essentially interdependent on each other. When a node fails, its neighbours can transmit information through their remaining viable neighbours. Therefore, a successful study of the multilayer infrastructure systems must simultaneously consider the connectivity links and the hidden interdependencies among their components.

In the previous models, interdependencies are always considered among the nodes across networks in multilayer networks or interdependent networks, and a node that is failing will be removed completely if any of its interdependent nodes in other layers are malfunctional. Although some works have explored the percolation model of networks with interdependency groups [51–56], the effects of topological properties of the network layer composed of interdependency links on the robustness of a multilayer network have not been studied explicitly due to the assumption that there is "strong" interdependence. Specifically, once a node fails, all the other nodes will be removed if they are connected in the same cluster by interdependency links, no matter the property of the topological structures. In some real-world complex network system, a node can still sustain partial functionality, and total destruction will not occur if its interdependency neighbours fail. Such examples in real world system are numerous, a failure node can not destroy its neighbours completely in a power network, and the bankruptcy of a financial institution can not collapse the other connected institutions in a financial network. In this paper, we adopt the "weak" interdependency assumption whereby the failure of a node can damage some links of its interdependency neighbours, but not necessarily give rise to the total loss of neighboring nodes and links [13,57].

It is of interest to study the robustness of multilayer networks with both connectivity and interdependency links, wherein each type of link represents the connectivity and interdependency relationships, respectively. By tuning the interdependency strength among nodes and the density of interdependency links, we find that the system can undergo first- order, second-order, and double phase transitions with a change in the initial preserved nodes. Our results indicate that the interdependency strength among nodes and the density of interdependency links have a great influence on the robustness of the system and the collapse manner in the case of random attack. We derive a theoretical framework to predict the characteristic variations in the nature of the phase transition as induced by the average degrees of connectivity links and the average degrees of interdependency links, with excellent numerical agreement based on the percolation dynamics in random networks and in scale-free networks. Our results have important implications for enhancing network robustness and resilience (as in protecting or designing a multilayer infrastructure system); properly controlling the interdependency strength and the density of interdependency links can be beneficial.

2. Model

We consider that a double-layer network composes of a number N of nodes and two types of links, one type of which is the connectivity links that compose the connectivity layer, and the other type of which is the interdependency links that compose the interdependency layer(See Fig. 1 for an illustration). Each node i ($i = 1, 2, \cdots N$) has a number k of connectivity links and a number r of interdependency links in the network. The integer numbers k and r follow the connectivity degree distribution $p(k)$ and the interdependency degree distribution $q(r)$, respectively. We begin by randomly removing a fraction, $1 - p$, of the nodes of the network(i.e., a fraction p of nodes are reserved). If a node i in the network fails, its connectivity links will be removed from the network completely. At the same time, each connectivity link of its neighbours in the interdependency layer will be disabled with a probability $1 - \alpha$, with $1 - \alpha$ characterizing the destructiveness in the neighbourhood of the failed node i. After that, some nodes will fail as a result of isolation from the giant component in the connectivity layer, which, in turn, will cause more links to be removed in the connectivity layer through the interdependency links. This recursive or cascading process proceeds until no failures occur, and the system will reach a stable state. In this work, we use the final size S of the giant components to evaluate the robustness of the system, as in previous works [8,13].

We devise our percolation model to be a complex networked system with hidden interdependencies among the nodes, which are represented by the interdependency links. The interdependency strength among nodes is characterised by parameter α, which governs the damage that a node will endure provided that one of its interdependency partners fail. When $\alpha \to 1$, the strength of the interdependence among nodes negligible so that, practically, failures cannot propagate from

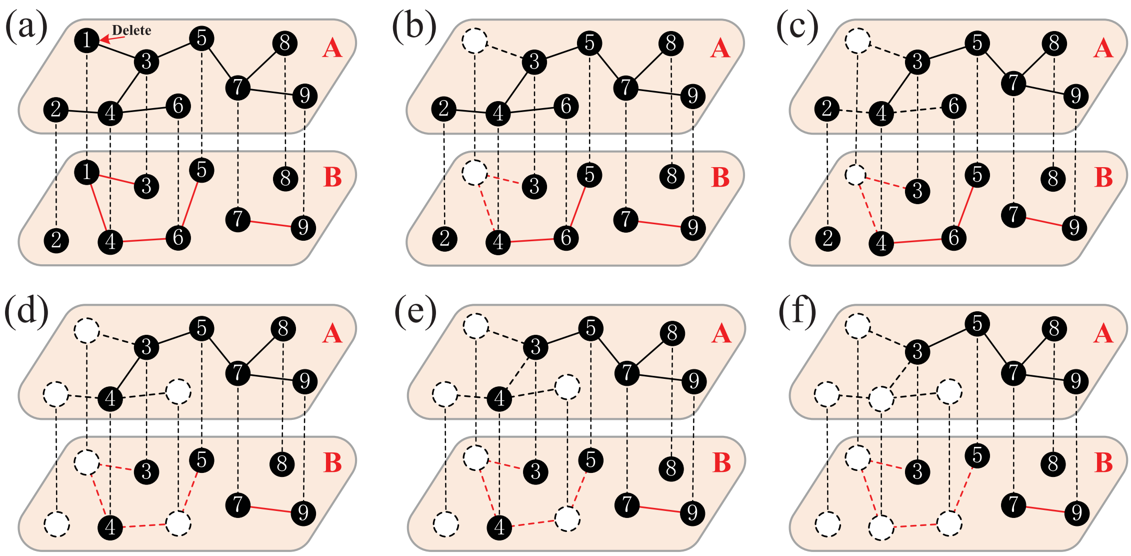

Fig. 1. Schematic illustration of the cascading process of a double-layer network. The system has two types of links, one of which is connectivity links (black straight line in layer A), and the other is interdependent links (red straight line in layer B). (a) Initial failure: node 1 is removed from the network. (b) The connectivity link of node 1 is removed from the network. (c) Node 4 is impacted, and two connectivity links are removed due to the interdependency between nodes 4 and 1. (d) Nodes 2 and 6 fail due to isolations from the giant component of the connectivity layer. (e) The link between nodes 4 and 3 is removed due to the interdependency between nodes 4 and 6. (f) Node 4 fails and the system reaches a steady state. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

one node to another through an interdependency link. In this case, our model reduces to the previous ordinary percolation model of a single network. The opposite limit $\alpha \to 0$ implies the maximum strength of interdependence. In this case, the failure of a node can destroy all the remaining nodes in the same connected component of the interdependency layer.

3. Theory

By using the connectivity degree distribution $p(k)$ and the interdependency degree distribution $q(r)$, the average connectivity degree $\langle k\rangle$ for connectivity layer and the average interdependency degree $\langle r\rangle$ for interdependency layer can be expressed as $\Sigma_k p(k)k$ and $\Sigma_r q(r)r$, respectively. In our study, we introduce a key parameter $\beta \equiv \frac{\sum_r q(r)r}{\sum_k p(k)k}$ to explore the impact of the relative density of interdependency links on the robustness of a multilayer system. We define the function $G_0(x) = \sum_k p(k)x^k$ as the generating function [58] that produces the degree distribution of random nodes in the connectivity layer, and $G_1(x) = \sum_k p(k)kx^{k-1}/\langle k\rangle$ is the generating function giving rise to the distribution for the number of outgoing links of randomly chosen links in the connectivity layer. In our study, we use the viable probability of a random node in the network to measure the final size S of the giant components since only the nodes in the final giant component are viable and a node has more probability to be viable if the final size S of the giant component is large [22,58]. When the system reaches a steady state, we use the self-consistent probabilistic approach to solve the viable probability S of a random node in the network. We define x to be the probability that a randomly chosen link in the connectivity layer belongs to the giant component, and thus each link of a random node is preserved with the probability $\alpha^t x$ if there are t failed nodes in its interdependency neighbourhood. Therefore the viable probability of node i of degree k is $1-(1-\alpha^t x)^k$. Considering the probability distribution of the connectivity degree k, we can derive the viable probability of a random node in the network according to the generating function G_0

$$S = p\sum_r q(r)\sum_t^r [1 - G_0(1-\alpha^t x)]f(t), \tag{1}$$

where $f(t)$ denotes the probability distribution of the number t of failed interdependency partners for a random node, and $q(r)$ is the interdependency degree distribution of a random node as defined in Section 2.

Similarly, we can also derive the equation for x according to the branching process in the connectivity layer. Following a randomly chosen connectivity link, we arrive at a node i of degree k, where k follows the distribution $kp(k)/\langle k\rangle$ since high-degree nodes have more links attached to them than that of low-degree ones [22,58]. The probability that node i has at least one outgoing link to the giant component is $1-(1-\alpha^t x)^{k-1}$ with t denoting the number of failed interdependency partners of node i. According to the probability distribution $f(t)$, the viable probability of a random link in the connectivity

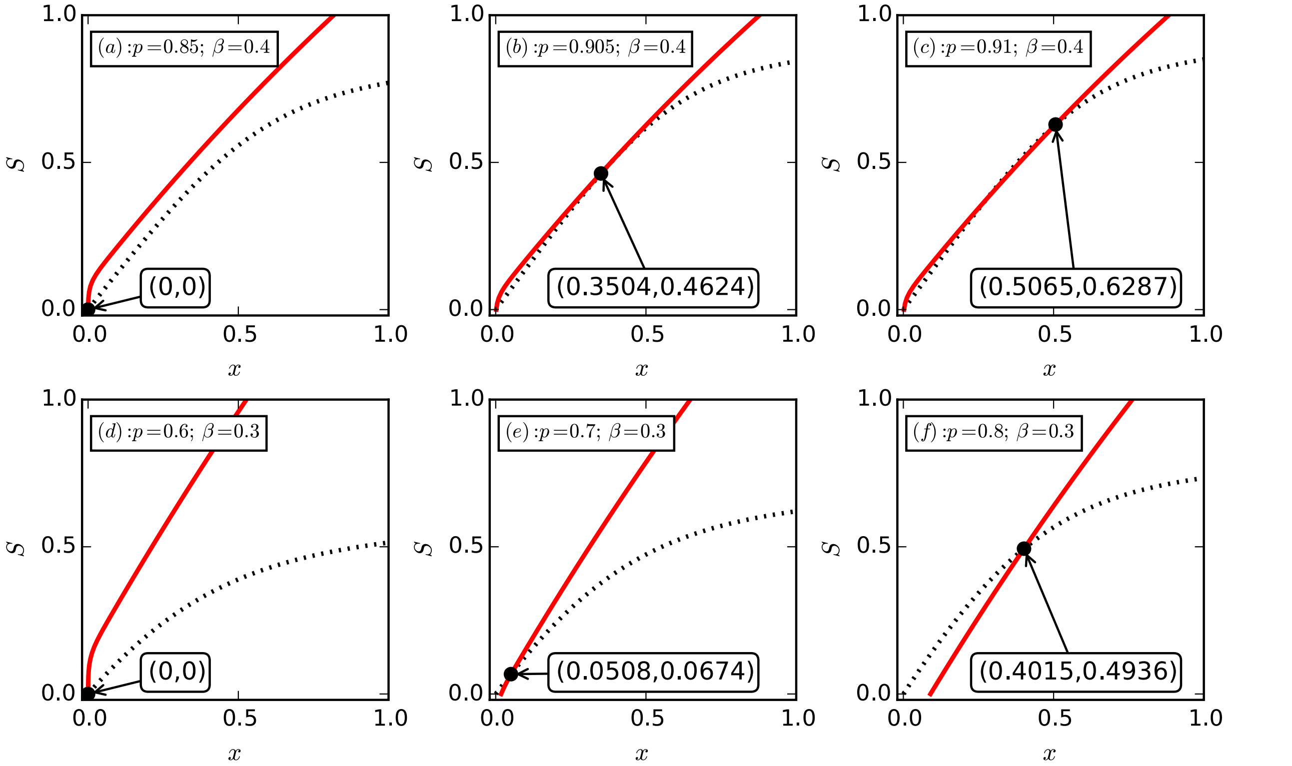

Fig. 2. Shown are the graphical solutions of Eqs. (1) and (2) for the different values of β and p, as marked by the black dots. (a-c) For $\beta = 0.4$, the results for $p = 0.85$, $p = 0.905$ and $p = 0.91$, respectively. (d-f) For $\beta = 0.3$, the solutions for $p = 0.6$, $p = 0.7$, and $p = .8$, respectively. The average connectivity degree is $\langle k\rangle = 4$ and $\alpha = 0.4$ for all panels. The black line denotes the function curve of Eq. (1) and the red line denotes the function curve of Eq. (2) for each panel. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

layer is

$$x = p\sum_{r} q(r)\sum_{t}^{r}\alpha^{t}[1 - G_1(1 - \alpha^{t}x)]f(t) \equiv F(S, x, p). \tag{2}$$

By using the viable probability S for a random node, we can write the probability distribution function $f(t)$ for a random node with r interdependency nodes

$$f(t) = \binom{r}{t}(1 - S)^{t}S^{r-t}. \tag{3}$$

Eqs. (1) and (2) can be solved numerically for a given set of the connectivity degree distribution $p(k)$ and the interdependency degree distribution $q(r)$. The size of the giant component of the network can emerge as a first-order(second-order) percolation transition manner as p goes through the percolation transition point $p_c^{I}(p_c^{II})$. When p approaches the second-order percolation transition point p_c^{II}, the probabilities x and S tend to continuously be zero. We can have the Taylor expansion of Eq. (2) for $x \to 0$:

$$x = F'(0, 0, p_c^{II})x + \frac{1}{2}F''(0, 0, p_c^{II})x^2 + O(x^3). \tag{4}$$

By dividing both sides of Eq. (4) by x, we have

$$F'(0, 0, p_c^{II}) + \frac{1}{2}F''(0, 0, p_c^{II})x + O(x^2) = 1. \tag{5}$$

and the nontrivial solution of Eq. (2) appears when $F'(0, 0, p_c^{II}) = 1$.

By taking the derivative of $F(S, x, p)$ with respect to x, we have

$$F'(0, 0, p_c^{II}) = p_c^{II}G_1'(1)\sum_{r} q(r)\alpha^{2r}, \tag{6}$$

and

$$p_c^{II} = \frac{1}{G_1'(1)\sum_r q(r)\alpha^{2r}}. \tag{7}$$

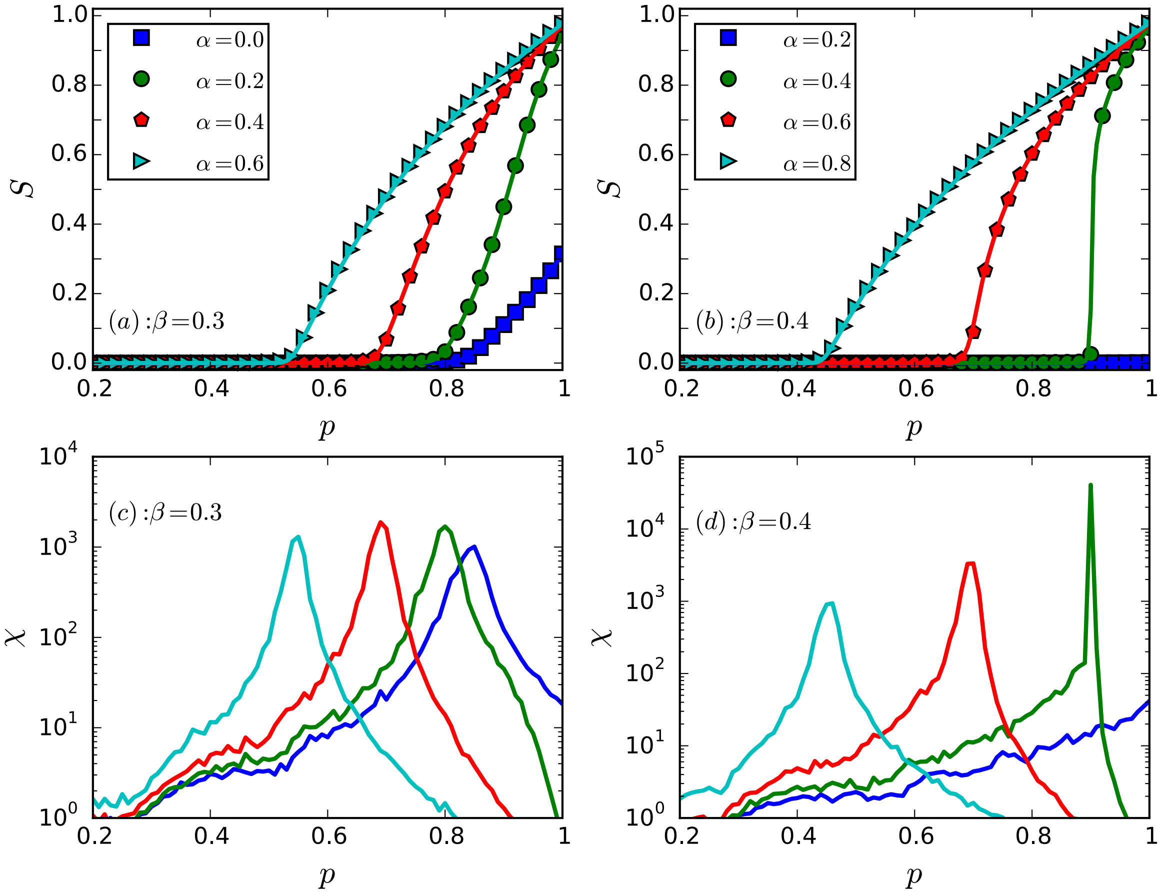

Fig. 3. Simulation results for percolation transitions on double-layer random networks. (a) and (b) are the fraction of nodes in the giant component at the end of cascade in the network, which is denoted by S, as a function of p for $\beta = 0.3$ and $\beta = 0.4$, respectively. (c) and (d) are the susceptibility as a function of p for $\beta = 0.3$ and $\beta = 0.4$, respectively. The results were obtained by averaging over 100 independent realizations, where the network size is $N = 10^5$ with the average degree $\langle k \rangle = 4$. The dotted lines behind the symbols denote the theoretical predictions that were obtained by Eqs. (1) and (2).

When $\alpha \to 1$, cascading failures do not occur as our model reduces to the ordinary percolation on a single network, and the percolation transition point $p_c^{II} = \frac{1}{G_1'(1)}$, which has been analytically derived in [20].

The probability x jumps from a nontrivial value x_c to zero abruptly at the first-order percolation transition point p_c^I. In that case, the nontrivial solution x_c of Eq. (2) satisfies

$$\frac{\partial F(S, x, p)}{\partial x}|_{x=x_c, p=p_c^I} = 1. \tag{8}$$

The first-order percolation transition point p_c^I can be solved numerically by Eqs. (1), (2) and (8).

The boundary between first- and second-order percolation transitions is determined by the conditions that the first- and second-order transitions are satisfied simultaneously, i.e., $p_c^I = p_c^{II} \equiv p_c$. In this case, the critical probability x_c tends to zero at the percolation point p_c. Substituting Eq. (7) into Eq. (4), we can have

$$\frac{1}{2}F''(0, 0, p_c^{II})x^2 + O(x^3) = 0, \tag{9}$$

and the condition that nontrivial x_c appears, which is $F''(0, 0, p_c^{II}) = 0$.

By taking the second derivative of $F(S, x, p)$ with respect to x, we can obtain that the critical value α_c, which separates the first- and second-order percolation transitions, and satisfies

$$2G_0'(1)\sum_{r\neq 0} q(r)r(\alpha^{2r-2} - \alpha^{2r})\frac{\sum_r q(r)\alpha^r}{\sum_r q(r)\alpha^{2r}} = G_1''(1)\sum_r q(r)\alpha^{3r}, \tag{10}$$

and

$$G_1'(1)\sum_r q(r)(\alpha_c)^{2r} \geq 1. \tag{11}$$

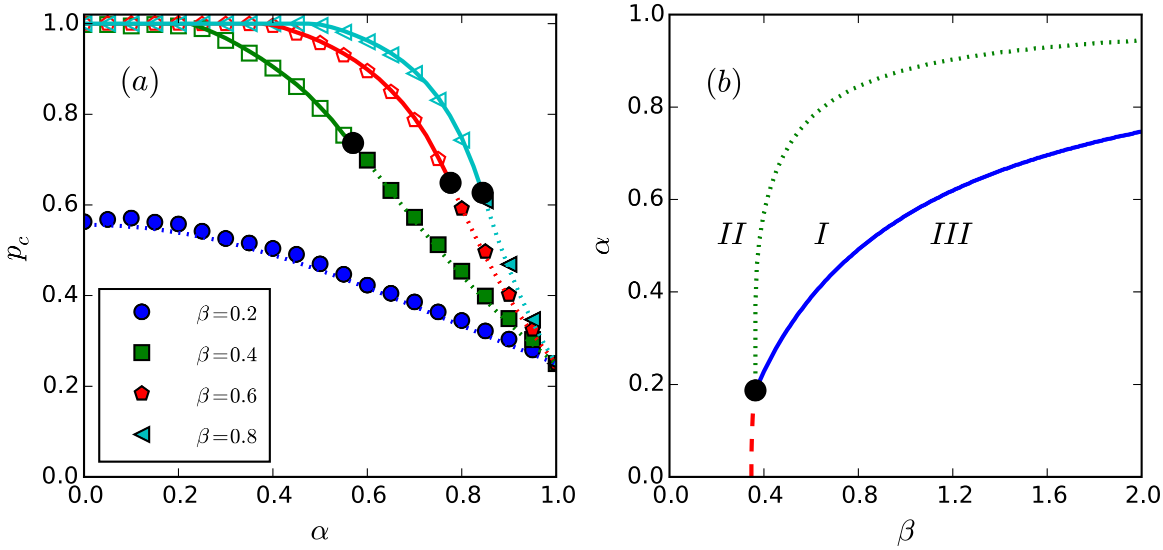

Fig. 4. (a) Dependence of the percolation transition point p_c on parameter α for double-layer random networks. For $\beta = 0.4$, 0.6 and 0.8, there exists a critical point α_c, which is marked by the solid dot, and divides the α interval into two subregions with first-order percolation transitions (solid line) and second-order percolation (dotted line) transitions, respectively. (b) The phase diagram for the double-layer random networks. The parameter plane is divided into three regions with numerals *I*, *II* and *III* denoting the parameter areas for first-, second-order percolation transitions and the unstable state, respectively. The simulation results are obtained by averaging over 100 independent realizations, where the network size is $N = 10^5$ with an average connectivity degree $\langle k \rangle = 4$.

4. Results

4.1. Random networks

We first study the percolation on a double-layer random network, where the nodal degree of the connectivity layer follows a Poisson distribution with an average $\langle k \rangle$ [59,60], and the nodal degree of interdependency layer also follows a Poisson distribution with an average $\langle r \rangle = \beta \langle k \rangle$. Fig. 2 displays the graphical solutions of Eqs. (1) and (2) for a double-layer random network with different parameter values of β and p. When $\beta = 0.4$ and $\alpha = 0.4$, the trivial solution at the point $(x = 0, S = 0)$ indicates that the network is completely fragmented for $p = 0.85$. While for $p = 0.905$, the solution at the tangent point (0.3504,0.4624) results in an abrupt change in both x and S, which signifies a first-order percolation transition. For $\beta = 0.3$ and $\alpha = 0.4$, the crossing point for x and S varies continuously from (0,0) to nontrivial values, thus characterizing a second-order percolation transition.

Fig. 3(a) and (b) show the size of the giant component in the network, which is denoted by S, versus the fraction p of initially reserved nodes for double-layer random networks for $\beta = 0.3$ and $\beta = 0.4$, respectively. For a lower value of β (e.g., $\beta = 0.3$), we can find that the size of the giant component for the network percolates continuously at a threshold p_c^{II} for all α. As β is increased (e.g., $\beta = 0.4$), the network can percolate in different manners depending on the value of α. When α is larger than the critical value α_c, the network percolates continuously (e.g., $\alpha = 0.6$ or $\alpha = 0.8$), and the network percolates discontinuously for $\alpha < \alpha_c$ (e.g., $\alpha = 0.4$). For a further decreased α(e.g., $\alpha = 0.2$), the network is unstable and the percolation transition point $p_c \to 1$, which means the system cannot percolate even if a negligible fraction of nodes is removed. Fig. 3(c) and (d) show the fluctuation χ of the giant component, i.e., the susceptibility, as functions of p for $\beta = 0.3$ and $\beta = 0.4$, respectively, from which we find that a peak in the susceptibility χ occurs at the percolation point. Therefore, we can use the peak position to estimate the percolation threshold [61]. These results indicate that both the density of the interdependency links and the interdependency strength among the nodes play important roles in determining the collapse manners of a double-layer random network. Theoretical predictions are also presented in Fig. 3, which agree with the simulation results very well.

Fig. 4 (a) shows the percolation transition points p_c ($p_c^I or p_c^{II}$) versus α for different values of β. We see that the network percolates continuously within the full range of α when $\beta = 0.2$. However for $\beta = 0.4$, 0.6 and 0.8, the network percolates continuously for $\alpha > \alpha_c$ and percolates discontinuously for $\alpha < \alpha_c$. When the value of α is further reduced, the network reaches an unstable state and the percolation transition point $p_c \to 1$. Fig. 4(b) shows the phase diagram for double-layer random networks, from which we can see that the parameter plane is divided into three regions by three curves. The boundary between the regions of the first-order and the second-order percolation transitions are determined by Eqs. 10 and 11, and the boundary between the regions of unstable stable and first(second)-order percolation transition is determined by imposing $p_c^I = 1 (p_c^{II} = 1)$. By using this result, we can predict which state a system will evolve into for different parameter

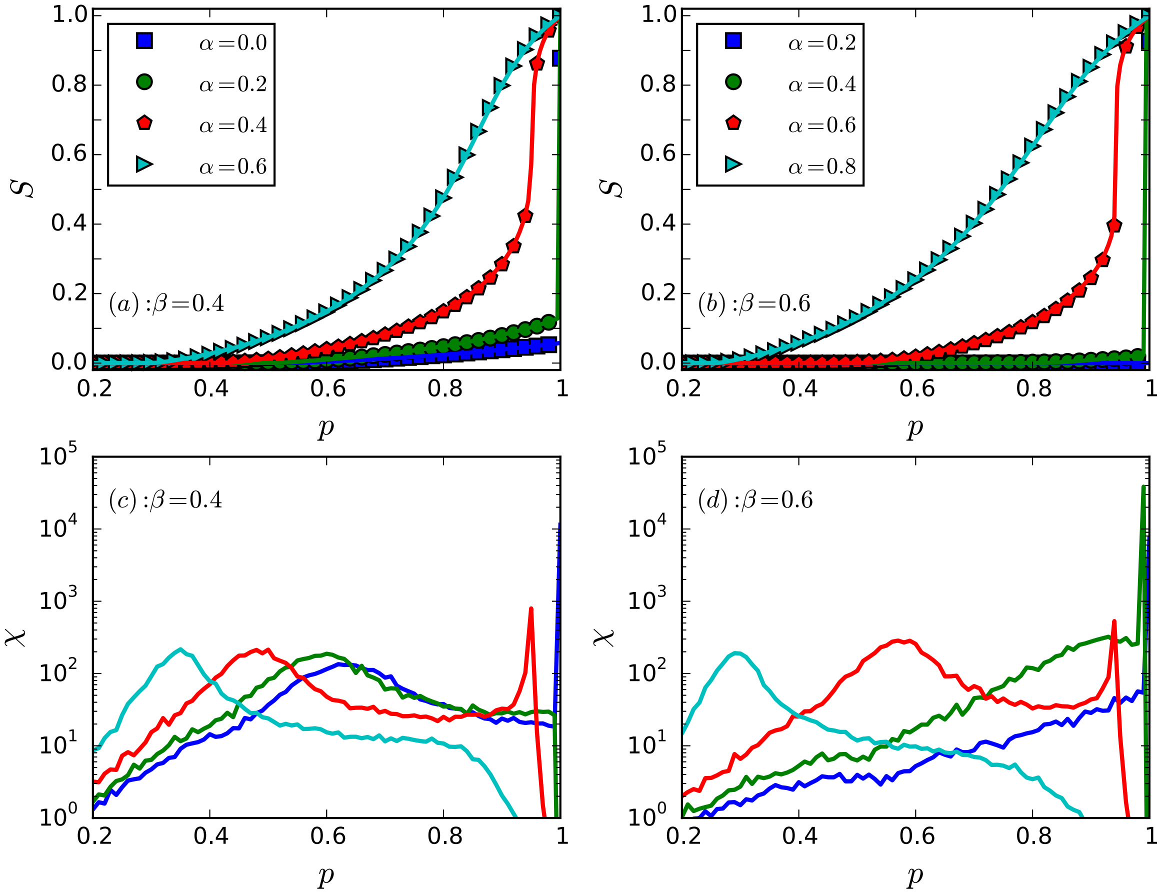

Fig. 5. Simulation results for percolation transitions on double-layer networks composed of a scale-free networked connectivity layer and a random networked interdependency layer. (a) and (b) are the fraction of nodes in the giant component at the end of the cascade in the network, denoted by S, as a function of p for $\beta = 0.4$ and $\beta = 0.6$, respectively. (c) and (d) are the susceptibility as a function of p for $\beta = 0.4$ and $\beta = 0.6$, respectively. These results are obtained by averaging over 100 independent realizations, where the network size is $N = 1 \times 10^5$ with the average connectivity degree $\langle k\rangle = 4$, $k_{\min} = 2$, $k_{\max} = 63$ and $\gamma = 2.5$, respectively. The dotted lines behind the symbols denote the theoretical predictions that were obtained by Eqs. (1) and (2).

settings. This result further proves that the robustness of the multilayer network and the collapse manner can be impacted greatly by both the density of interdependency links and the interdependency strength among nodes.

4.2. Scale-free networks

Next, we study the percolation on a double-layer network, where the nodal degree of the connectivity layer follows a truncated pow-law distribution with an average $\langle k\rangle$, and the nodal degree of interdependency layer follows a Poisson distribution with an average $\langle r\rangle = \beta\langle k\rangle$. For the connectivity layer, the degree distribution is $p(k) \sim k^{-\gamma}\,(k_{\min} \le k \le k_{\max})$, where $k_{\min}$ and $k_{\max}$ are the lower and upper bounds of the degree, respectively, and γ is the power law exponent. Fig. 5(a) and (b) show percolation transitions on a double-layer network composed of a scale-free networked connectivity layer and a random networked interdependency layer for $\beta = 0.4$ and $\beta = 0.6$, respectively. When $\beta = 0.4$, we can see that the network first percolates as a continuous manner and then exhibits a discontinuous phase transition with the increase of p for some values of α (e.g., $\alpha = 0$, 0.2, and 0.4). This result indicates the existence of a double-phase transition, which is characterized by the double peak phenomenon of the susceptibility versus p. The double phase transition signals that there are two stages in the collapse process [61,62]. First, a large fraction of nodes fail simultaneously and suddenly in the network, and then the remaining active nodes fail in a continuous manner. This result is in concordance with the phenomena of cascading failures occurred in the critical infrastructure systems, whereby a large-scale of nodes fail abruptly and unexpectedly and only a small fraction of functional nodes are left. For an increased α (e.g., $\alpha = 0.6$), the double-phase transition becomes a single continuously percolation transition. When $\beta = 0.6$, we can find that the double phase transition is still persistent. The double-phase transition can change to a first-order percolation transition for some decreased values of α (e.g., $\alpha = 0.2$), or it can change to a second-order percolation transition for some increased values of α (e.g., $\alpha = 0.8$).

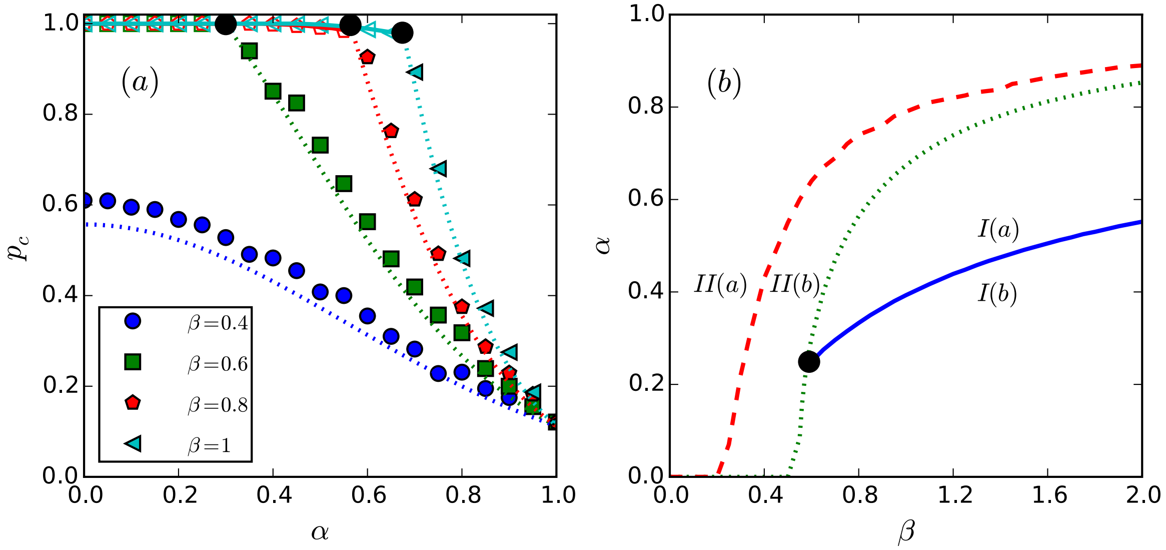

Fig. 6. (a) Dependence of the percolation transition point p_c on parameter α. For $\beta = 0.4$, 0.6 and 1, there exists a critical point α_c, which is marked by the solid dot, which divides the α interval into two subregions of first-order percolation transition (solid line) and second-order percolation (dotted line) transition, respectively. (b) The phase diagram for double-layer random networks. The parameter plane is divided into first- and second-order percolation transition regions. The region of first-order percolation transition is divided into a stable subregion ($I(a)$) and an unstable subregion ($I(b)$), and the region of second-order percolation transition is divided into a single phase transition subregion $II(a)$ and a double phase transition subregion $II(b)$. The simulation results are obtained by averaging over 100 independent realizations, where the network size is $N = 10^5$ with the average connectivity degree $\langle k\rangle = 4$, $k_{\min} = 2$, $k_{\max} = 63$ and $\gamma = 2.5$, respectively.

Fig. 6(a) shows the percolation transition points p_c as functions of α for different values of β. For $\beta = 0.4$, the networks percolate continuously for any value of α, and for $\beta = 0.6$, 0.8 and 1, the manners of percolation transition are separated discontinuously and continuously by a critical value of α_c. Fig. 6(b) shows the phase diagram for the double-layer networks on the $\alpha - \beta$ plane, which is divided into the regions of first- and second-order percolation transition by the curve α_c versus β. The region of the first-order percolation transition is further divided into a stable subregion ($I(a)$) and an unstable subregion ($I(b)$), where the boundary is determined by the condition $p^I = 1$. In addition, the region of the second-order percolation transition is further divided into a single-phase transition subregion $II(a)$ and a double-phase transition subregion $II(b)$, where the boundary is determined by the presence or absence of the tangent point in the solution plane as shown in Fig. 2.

4.3. Empirical networks

We demonstrate our model on a double-layer empirical network, which is composed by an autonomous system-level Internet and the power grid of the western states of the USA (data sets available at http://konect.uni-koblenz.de/). The autonomous system-level Internet consists of 6474 nodes [63] and the power grid has 4941 nodes with each being either a generator, a transformer or a substation [64]. We randomly choose a number of nodes from the autonomous system-level Internet as the corresponding partners of the nodes in the power grid, and define an interlink between a power grid node and an Internet node until all the selected Internet nodes and the power grid nodes are connected, whereby the interlink denotes the exchange of electricity and communication services between them. According to the scheme of our model, if a node in the power grid fails, the Internet node that interconnects to it will fail too, and vice versa. At the same time, the neighbours of the failed node in the power grid will suffer a loss of some links in the Internet as the redistribution of electrical current or voltage.

Fig. 7 shows the size S of the giant component of the double-layer network versus p for different values of α. For some small values of α (e.g., $\alpha \leq 0.5$), the size of the giant component reduces drastically and abruptly as p is decreased from one, and then the size of the giant component reduces gradually. For some large values of α (e.g., $\alpha \geq 0.6$), the abrupt change of S disappears and the size of the giant component changes continuously with the decrease of p. These results are consistent with that of synthetic networks and the observations of cascading failures occurred in the critical infrastructure systems, which demonstrates that an appropriate interdependency strength can be quite beneficial for the preventing sudden, system-wide failures.

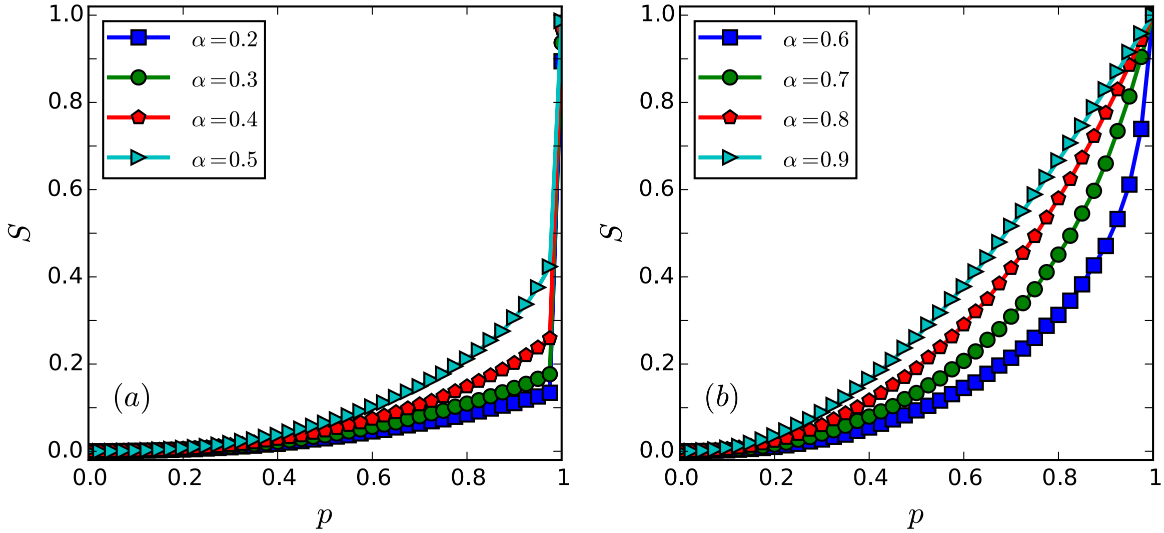

Fig. 7. The size S of the giant component of the double-layer network versus p. (a) shows the results for $\alpha = 0.2,\ 0.3,\ 0.4$ and 0.5, respectively and (b) shows the results for $\alpha = 0.6,\ 0.7,\ 0.8$ and 0.9, respectively.

5. Conclusion

The multilayer networks always have different types of links, which will act as different channels for the spread of failures when it is suffering attack. Therefore, the coaction of these links with different properties will ineluctably exert an important impact on the dynamical processes of the whole system. In most of the existing works on the robustness of multilayer networks, the interactions among nodes are always treated as essentially homogeneous and represented by connectivity links. However, the interactions among nodes in different layers are always heterogeneous in nature for real-world multilayer networks. In this work, we studied a class of multilayer networks with both interdependent links and connectivity links, and we found that rich phase transition phenomena occurred with the varying density of interdependency links and interdependency strengths among nodes. In particular, we found that a double-layer system with a scale-free distribution of connectivity degrees can collapse in a two-stage manner: a large fraction of nodes fails suddenly, and then the remaining active nodes fail gradually. We further validated this result by using a double-layer system of an Internet and a power grid with randomly assigned one-to-one correspondences. This result is in concordance with the phenomena of cascading failures that occurred in the critical infrastructure systems, thus, it gives a possible qualitative explanation for an unexpected large-scale damage or disruptive avalanche in real-world infrastructure systems. In addition, our results also prove that the robustness of the multilayer networks and the collapse manner can be significantly impacted by both the density of the interdependency links and the interdependency strength among nodes. Our model provides a feasible way to depict an important intrinsic quality of the multilayer networks, which is the topological structure of hidden interdependencies among nodes. Therefore, it is possible to study the effects of additional topological properties of the interdependent layer on multilayer networks. Thus, the method presented here opens the possibility for understanding the robustness of the real-world system and affords new insights for the design of interdependent systems.

Declaration of Competing Interest

The authors declare that they have no conflicts of interest.

CRediT authorship contribution statement

Yan-Yun Cao: Investigation, Visualization, Software, Validation. **Run-Ran Liu:** Methodology, Writing - original draft, Formal analysis, Funding acquisition. **Chun-Xiao Jia:** Conceptualization, Writing - review & editing. **Bing-Hong Wang:** Conceptualization, Resources, Funding acquisition.

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