Applied Mathematics Letters 116 (2021) 107017

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

# On the 3D ideal MHD equations with partial damping

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ARTICLE INFO

*Article history:*
Received 14 December 2020
Accepted 31 December 2020
Available online 6 January 2021

*Keywords:*
MHD
Damping
Global solution
Energy cascade

ABSTRACT

We consider the 3D ideal MHD equations on $\mathbf{T}^3$ with partial dampings. The dampings, which act as partial dissipations, are defined by the fractional Laplacian removing Fourier modes in certain symmetric subsets $K_j \subset \mathbf{Z}^3$ $(j = 1, 2)$. This model is motivated by Elgindi et al. (2017) and Kim and Dubrulle (2002) where the phenomenon of energy cascade is analyzed for the 2D inviscid fluid flow with partial dampings. Our main result shows that this model has a unique global-in-time smooth solution under suitable assumptions on $K_j$ and $\alpha_j$.


## 1. Introduction

We consider the Cauchy problem of the 3D ideal MHD equations with partial dampings:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu Y_1^{\alpha_1} u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b - b \cdot \nabla u &= \mu Y_2^{\alpha_2} b, \\ div\, u = div\, b &= 0, \\ u|_{t=0} = u_0,\ b|_{t=0} &= b_0, \end{aligned} \tag{1}$$

where $u = u(x,t) : \mathbf{T}^3 \times [0,T) \to \mathbf{R}$ is the fluid velocity, $b = b(x,t) : \mathbf{T}^3 \times [0,T) \to \mathbf{R}$ is the magnetic field and $p = p(x,t) : \mathbf{T}^3 \times [0,T) \to \mathbf{R}$ is the pressure. The initial data $(u_0, b_0)$ is given. The constants $\nu > 0$ and $\mu > 0$ denote the viscosity and magnetic diffusivity, respectively. The damping terms are given by the operators $Y_j^{\alpha_j}$ $(j = 1, 2)$, which are of the form:

$$\widehat{Y_j^{\alpha_j} h}(k) = -|k|^{2\alpha_j} \widehat{h}(k) \mathbf{I}_{K_j^c}, \tag{2}$$

where $\widehat{h}(k) = \int_{\mathbf{T}^3} h(x) e^{-ik\cdot x}\, dx$ is the Fourier transform of $h = h(x)$, $K_j \subset \mathbf{Z}^3$ are symmetric subsets, $K_j^c = \mathbf{T}^3 \setminus K_j$ and $\mathbf{I}_X$ is the characteristic function taking value unit if $k \in X$ otherwise zero.

Observe that if $K_1 = K_2 = \emptyset$, then (1) is the 3D fractional MHD equations (see, e.g. [1,2]), which is a natural generalization of the classical 3D viscous MHD equations ($\alpha_1 = \alpha_2 = 1$). It is known that when

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https://doi.org/10.1016/j.aml.2021.107017

$\alpha_1 + \alpha_2 \geq \frac{5}{2}$ and $1 \leq \alpha_2 \leq \frac{5}{4} \leq \alpha_1 < \frac{5}{2}$, then the 3D fractional MHD has a unique global-in-time smooth solution whenever the initial data $(u_0, b_0) \in H^3(\mathbf{T}^3)$. On the other hand, the global existence of a unique smooth large solution to the 3D ideal MHD or to the 3D viscous-diffusive MHD equations is still a big open problem. Moreover, the case when $\nu = 0$ or $\eta = 0$ in the 3D viscous-diffusive MHD equations is also open since it is worse than the viscous-diffusive case. Therefore, it should be interesting to ask what will happen when we remove certain Fourier modes in the operators $(-\Delta)^{\alpha_j}$ when $\alpha_j \geq \frac{5}{4}$ (Lions' component, see [3]). The model (1) is also physically interesting when we consider the energy cascade phenomenon in 3D turbulence. For this regard, we refer to [4,5] and references therein.

The aim of this paper is to prove the global existence of smooth solutions under suitable assumptions on undamped sets $K_j$ and on the indices $\alpha_j$ $(j = 1, 2)$, while in the near future we will consider the dynamic behavior of such global smooth solutions by tools developed in [4–6].

To be precise let us give the standard notation of Fourier series:

$$h(x) = \frac{1}{(2\pi)^3} \sum_{k \in \mathbf{Z}^3} \hat{h}(k) e^{ik\cdot x}. \tag{3}$$

Moreover, we define the operators $Z_j^{\alpha_j}$ $(j = 1, 2)$ by

$$\widehat{Z_j^{\alpha_j} h}(k) = -|k|^{2\alpha_j} \widehat{h}(k) \mathbf{I}_{K_j}, \tag{4}$$

so that $Y_j^{\alpha_j} + Z_j^{\alpha_j} = (-\Delta)^{\alpha_j}$. Clearly, both $Y_j^{\alpha_j}$ and $Z_j^{\alpha_j}$ are nonnegative and thus we can define $Y_j^{\alpha_j/2}$ and $Z_j^{\alpha_j/2}$ by

$$\widehat{Y_j^{\alpha_j/2}} h(k) = -|k|^{\alpha_j} \widehat{h}(k) \mathbf{I}_{K_j^c} \text{ and } \widehat{Z_j^{\alpha_j/2}} h(k) = -|k|^{\alpha_j} \widehat{h}(k) \mathbf{I}_{K_j}, \tag{5}$$

respectively. We will also use the operators $\Lambda^{2\alpha_j} = (-\Delta)^{\alpha_j}$ which are defined by (see, e.g. [7])

$$\widehat{\Lambda^{2\alpha_j} h}(k) = -|k|^{2\alpha_j} \widehat{h}(k) \tag{6}$$

In order to simplify the statement below, we denote $H^3_{div}(\mathbf{T}^3) = \{u \in H^3(\mathbf{T}^3) \mid div\, u = 0\}$.

Now our main results can be summarized as follows.

If $\alpha_1 = \alpha_2$, we have

**Theorem 1.1.** *Let $K_j \subset \mathbf{Z}^3$ $(j = 1, 2)$ be symmetric and finite. If $\alpha_1 = \alpha_2 \geq \frac{5}{4}$ and $(u_0, b_0) \in H^3_{div}(\mathbf{T}^3)$, then (1) has a unique global-in-time smooth solution.*

If $\alpha_1 \neq \alpha_2$, we have

**Theorem 1.2.** *Let $K_j \subset \mathbf{Z}^3$ $(j = 1, 2)$ be symmetric and finite. If $1 \leq \alpha_2 \leq \frac{5}{4} < \alpha_1 < \frac{5}{2}$, $\alpha_1 + \alpha_2 \geq \frac{5}{2}$ and $(u_0, b_0) \in H^3_{div}(\mathbf{T}^3)$, then (1) has a unique global-in-time smooth solution.*

**Remark 1.3.** It should be interesting to consider the case when $K_1$ is finite and $K_2$ is co-finite with $\alpha_1, \alpha_2 > \frac{5}{4}$ (no matter how large). We do not know so far whether (1) under this assumption has a global-in-time smooth solution or not.

Before proving Theorems 1.1 and 1.2, let us first introduce the notion of a global weak solution to (1). Let $\mathcal{H}^{\alpha_j} = \{u \in L^2(\mathbf{T}^3) \mid Y_j^{\alpha_j} u \in L^2(\mathbf{T}^3)\}$ endowed with the norms $\|u\|_{\mathcal{H}^{\alpha_j}} = \|u\|_{L^2(\mathbf{T}^3)} + \|Y_j^{\alpha_j} u\|_{L^2(\mathbf{T}^3)}$, $j = 1, 2$. Clearly, $\mathcal{H}^{\alpha_j}$ are Banach spaces.

**Definition 1.4.** A pair $(u, b)$ is called a *weak solution* to (1), provided $(u, b)$ satisfies the following conditions:

(i) $u \in L^\infty(0, T; L^2) \cap L^2(0, T; \mathcal{H}^{\alpha_1/2})$, $b \in L^\infty(0, T; L^2) \cap L^2(0, T; \mathcal{H}^{\alpha_2/2})$.
(ii) $(u, b)$ satisfies (1) in the sense of distributions;
(iii) The energy inequality holds: for a.e, $0 \le t \le T$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \nu\|Y_1^{\alpha_1/2}u(t')\|_{L^2}^2 + \mu\|Y_2^{\alpha_2/2}b(t')\|_{L^2}^2\, dt' \le \|u_0\|_{L^2}^2 + \le \|b_0\|_{L^2}^2. \tag{7}$$

A weak solution is called a *strong solution* if in addition $u \in L^\infty(0, T; H^1) \cap L^2(0, T; \mathcal{H}^{\frac{\alpha_1+1}{2}})$, $b \in L^\infty(0, T; H^1) \cap L^2(0, T; \mathcal{H}^{\frac{\alpha_2+1}{2}})$. Clearly, a strong solutions is smooth and unique in the class of weak solutions. Thus, in Theorems 1.1 and 1.2 it suffices to show $(u, b)$ is strong.

**Theorem 1.5.** *Let $K_j \subset \mathbf{Z}^3$ $(j = 0,\, 1)$ be symmetric. For any $(u_0, b_0) \in H^3(\mathbf{T}^3)$, (1) has at least one global weak solution $(u, b)$.*

**Proof.** The proof is standard. We consider the following regularization model of (1) by introducing a sufficiently small positive parameter $\varepsilon$:

$$\begin{aligned} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon &= \nu Y_1^{\alpha_1} u^\varepsilon + \varepsilon Z_1^{\alpha_1} u^\varepsilon + b^\varepsilon \cdot \nabla b^\varepsilon, \\ \partial_t b^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon - b^\varepsilon \cdot \nabla u^\varepsilon &= \mu Y_2^{\alpha_2} b^\varepsilon + \varepsilon Z_2^{\alpha_2} b^\varepsilon, \\ div\, u^\varepsilon = div\, b^\varepsilon &= 0, \\ u^\varepsilon|_{t=0} = u_0,\ b^\varepsilon|_{t=0} &= b_0. \end{aligned} \tag{8}$$

Fix $\varepsilon > 0$. Assume that $(u^\varepsilon, b^\varepsilon)$ is the smooth solution to (8) with the initial data $(u_0, b_0) \in H^3(\mathbf{T}^3)$. Then, by taking into account the divergence-free condition and applying the standard energy estimates, we have $(u^\varepsilon, b^\varepsilon) \in C([0, \infty), L^2) \cap L^2([0, T); H^1)$ and

$$\begin{aligned} \frac{1}{2}(\|u^\varepsilon(t)\|_{L^2}^2 + \|b^\varepsilon(t)\|_{L^2}^2) + \int_0^t \nu\|Y_1^{\frac{\alpha_1}{2}} u^\varepsilon(t')\|_{L^2}^2 + \varepsilon\|Z_1^{\frac{\alpha_1}{2}} u(t')^\varepsilon\|_{L^2}^2 dt' \\ + \int_0^t \eta\|Y_2^{\frac{\alpha_2}{2}} b^\varepsilon(t')\|_{L^2}^2 + \varepsilon\|Z_2^{\frac{\alpha_2}{2}} b^\varepsilon(t')\|_{L^2}^2 dt' \\ = \frac{1}{2}(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2) \end{aligned} \tag{9}$$

for each $t \ge 0$.

Then, the existence of a global weak solution follows from the compactness argument and the Aubin–Lions lemma (see, e.g. [8,9]). This completes the proof.

The following two lemmas will be used in the sequel.

**Lemma 1.6.** *Let $K_j$ $(j = 1, 2)$ be symmetric and finite. Then, we have*

$$\begin{aligned} \|\Lambda^{\alpha_j} h\|_{L^2}^2 &\le \|Y_j^{\frac{\alpha_j}{2}} h\|_{L^2}^2 + \kappa_j^{2\alpha_j}\|h\|_{L^2}^2, \\ \|\Lambda^{\alpha_j+1} h\|_{L^2}^2 &\le \|Y_j^{\frac{\alpha_j+1}{2}} h\|_{L^2}^2 + \kappa_j^{2\alpha_j}\|\nabla h\|_{L^2}^2, \end{aligned} \tag{10}$$

*where $\kappa_j = \max_{k \in K_j} |k|$.*

**Proof.** We only prove the first inequality since the second one is similar. By the definition of $Y_j^{\alpha_j}$ and Plancherel's theorem, we have

$$\begin{aligned}\|\Lambda^{\alpha_j}h\|_{L^2}^2 &= \|\widehat{\Lambda^{\alpha_j}h}\|_{L^2}^2\\ &= \sum_{k\in\mathbf{Z}^3}|k|^{2\alpha_j}\big|\widehat{h}(k)\big|^2\\ &= \sum_{k\in K_j^c}|k|^{2\alpha_j}\big|\widehat{h}(k)\big|^2+\sum_{k\in K_j}|k|^{2\alpha_j}\big|\widehat{h}(k)\big|^2\\ &= \left\|\widehat{Y_j^{\frac{\alpha_j}{2}}h}\right\|_{L^2}^2+\kappa_j^{2\alpha_j}\|\hat{h}\|_{L^2}^2\\ &= \|Y_j^{\frac{\alpha_j}{2}}h\|_{L^2}^2+\kappa_j^{2\alpha_j}\|h\|_{L^2}^2.\end{aligned}\tag{11}$$

**Lemma 1.7.** *Let $K_j$ $(j=1,2)$ be symmetric and finite. Then, for any $\alpha_j\geq\frac{1}{2}$, there exists a constant $C>0$ such that*

$$\|h\|_{L^3}\leq C\|h\|_{L^2}^{\frac{2\alpha_j-1}{2\alpha_j}}\|Y_j^{\frac{\alpha_j}{2}}h\|_{L^2}^{\frac{1}{2\alpha_j}}+C\kappa_j^{\frac{1}{2}}\|h\|_{L^2}.\tag{12}$$

**Proof.** Applying Gagliardo–Nirenberg inequality and using Lemma 1.6, we have

$$\begin{aligned}\|h\|_{L^3} &\leq C\|h\|_{L^2}^{\frac{2\alpha_j-1}{2\alpha_j}}\|\Lambda^{\alpha_j}h\|_{L^2}^{\frac{1}{2\alpha_j}}\\ &\leq C\|h\|_{L^2}^{\frac{2\alpha_j-1}{2\alpha_j}}\left(\|Y_j^{\frac{\alpha_j}{2}}h\|_{L^2}+\kappa_j^{\alpha_j}\|h\|_{L^2}\right)^{\frac{1}{2\alpha_j}}\\ &\leq C\|h\|_{L^2}^{\frac{2\alpha_j-1}{2\alpha_j}}\|Y_j^{\frac{\alpha_j}{2}}h\|_{L^2}^{\frac{1}{2\alpha_j}}+C\kappa_j^{\frac{1}{2}}\|h\|_{L^2}.\end{aligned}\tag{13}$$

## 2. Proof of Theorem 1.1

Multiplying the first equation of (1) by $\Delta u$ and taking into account the divergence-free condition $\nabla\cdot u=0$, we can obtain the following identity by integration by parts

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|Y_1^{\frac{\alpha_1}{2}}\nabla u\|_{L^2}^2\\ =&-\int_{\mathbf{T}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,dx+\int_{\mathbf{T}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j\,dx\\ &-\int_{\mathbf{T}^3}b_k\cdot\partial_k\partial_i u_j\cdot\partial_i b_j\,dx.\end{aligned}\tag{14}$$

Similarly, multiplying the second equation of (1) by $\Delta b$ yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2+\|Y_2^{\frac{\alpha_2}{2}}\nabla b\|_{L^2}^2\\ =&-\int_{\mathbf{T}^3}\partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j\,dx+\int_{\mathbf{T}^3}\partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j\,dx\\ &+\int_{\mathbf{T}^3}b_k\cdot\partial_k\partial_i u_j\cdot\partial_i b_j\,dx.\end{aligned}\tag{15}$$

Then, combining (14) and (15) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\|Y_1^{\frac{\alpha_1}{2}}\nabla u\|_{L^2}^2+\|Y_2^{\frac{\alpha_2}{2}}\nabla b\|_{L^2}^2\\
=&-\int_{\mathbf{T}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,dx+\int_{\mathbf{T}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j\,dx\\
&-\int_{\mathbf{T}^3}\partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j\,dx+\int_{\mathbf{T}^3}\partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j\,dx\\
=&\,I+II+III+IV.
\end{aligned}\tag{16}$$

Next, we estimate $I$–$IV$ one by one.

$$\begin{aligned}
|I|=&\left|\int_{\mathbf{T}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,dx\right|\\
=&\left|\int_{\mathbf{T}^3}u_k\cdot(\partial_i\partial_k u_j\cdot\partial_i u_j+\partial_k u_j\cdot\partial_i\partial_i u_j)\,dx\right|\\
\le&\, C\|u\|_{L^3}\|\nabla u\|_{L^6}\|\varDelta u\|_{L^2}\\
\le&\, C\|u\|_{L^3}\|\nabla u\|_{L^2}^{\frac{1}{\alpha_1}}\|\varLambda^{\alpha_1+1}u\|_{L^2}^{\frac{\alpha_1-1}{\alpha_1}}\|\nabla u\|_{L^2}^{\frac{1}{\alpha_1}}\|\varLambda^{\alpha_1+1}u\|_{L^2}^{\frac{\alpha_1-1}{\alpha_1}}\\
\le&\,\frac{1}{2}\|\varLambda^{\alpha_1+1}u\|_{L^2}^2+C\|u\|_{L^3}^{\frac{\alpha_1}{\alpha_1-1}}\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{17}$$

Likewise, one can obtain

$$\begin{aligned}
|II|+|III|+|IV|\le&\, C\|u\|_{L^3}\|\nabla b\|_{L^6}\|\varDelta b\|_{L^2}\\
\le&\, C\|u\|_{L^3}\|\nabla b\|_{L^2}^{\frac{1}{\alpha_2}}\|\varLambda^{\alpha_2+1}b\|_{L^2}^{\frac{\alpha_2-1}{\alpha_1}}\|\nabla b\|_{L^2}^{\frac{1}{\alpha_2}}\|\varLambda^{\alpha_2+1}b\|_{L^2}^{\frac{\alpha_2-1}{\alpha_2}}\\
\le&\,\frac{1}{2}\|\varLambda^{\alpha_2+1}b\|_{L^2}^2+C\|u\|_{L^3}^{\frac{\alpha_2}{\alpha_2-1}}\|\nabla b\|_{L^2}^2.
\end{aligned}\tag{18}$$

By Lemma 1.6, we have

$$\|\varLambda^{\alpha_1+1}u\|_{L^2}^2+\|\varLambda^{\alpha_2+1}b\|_{L^2}^2\le\|Y_1^{\frac{\alpha_1+1}{2}}u\|_{L^2}^2+\|Y_2^{\frac{\alpha_2+1}{2}}b\|_{L^2}^2+\kappa_1^{2\alpha_1}\|\nabla u\|_{L^2}^2+\kappa_2^{2\alpha_2}\|\nabla b\|_{L^2}^2.\tag{19}$$

Moreover, by Lemma 1.7 and in view of the fact that $\|u\|_{L^2}\le\|u_0\|_{L^2}$, we have

$$\begin{aligned}
\|u\|_{L^3}^{\frac{\alpha_j}{\alpha_j-1}}\le&\, C\left(\|u\|_{L^2}^{\frac{2\alpha_j-1}{2\alpha_j}}\|Y_j^{\frac{\alpha_j}{2}}u\|_{L^2}^{\frac{1}{2\alpha_j}}+\kappa_j^{\frac{1}{2}}\|u\|_{L^2}\right)^{\frac{\alpha_j}{\alpha_j-1}}\\
\le&\, C\left(\|Y_j^{\frac{\alpha_j}{2}}u\|_{L^2}^{\frac{1}{2\alpha_j}}+\kappa_j^{\frac{1}{2}}\right)^{\frac{\alpha_j}{\alpha_j-1}}.
\end{aligned}\tag{20}$$

Combining all these estimates together, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\|Y_1^{\frac{\alpha_1}{2}}\nabla u\|_{L^2}^2+\|Y_2^{\frac{\alpha_2}{2}}\nabla b\|_{L^2}^2\\
\le&\, C\sum_{j=1}^{2}(\|Y^{\frac{\alpha_j}{2}}u\|_{L^2}^{\frac{1}{2(\alpha_j-1)}}+\kappa_j^{2\alpha_j}+k_j^{\frac{\alpha_j}{2(\alpha_j-1)}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)
\end{aligned}\tag{21}$$

Then, thanks to Gronwall's inequality and the energy inequality, one can obtain that

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla b(\cdot,t)\|_{L^2}^2\right)+\int_0^T\|Y_1^{\frac{\alpha_1}{2}}\nabla u\|_{L^2}^2+\|Y_2^{\frac{\alpha_2}{2}}\nabla b\|_{L^2}^2\,dt\\
\le&\, C\left(\|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2\right),
\end{aligned}\tag{22}$$

which completes the proof.

## 3. Proof of Theorem 1.2

Similar to the proof of Theorem 1.1, one can obtain the following estimates,

$$|I| \leq \frac{1}{2}\|\Lambda^{\alpha_1+1}u\|_{L^2}^2 + C\|\Lambda^{\alpha_1}u\|_{L^2}^{\frac{4\alpha_1}{6\alpha_1-5}}\|\nabla u\|_{L^2}^2, \tag{23}$$

where we used Hölder's, Gagliardo–Nirenberg's and Young's inequalities. Moreover, we have

$$|II| + |III| + |IV| \leq \frac{1}{2}\|\Lambda^{\alpha_2+1}b\|_{L^2}^2 + C\|\Lambda^{\alpha_1}u\|_{L^2}^{\frac{\theta_1}{1-\theta_2}}\|\nabla b\|_{L^2}^2, \tag{24}$$

where $\theta_1$, $\theta_2 \in [0,1]$, $\frac{\theta_1}{1-\theta_2} \leq 2$ and

$$\begin{aligned} &\frac{1}{a_1} + \frac{2}{a_2} = 1, \\ &\frac{1}{a_1} - \frac{1}{3} = \frac{1}{2}(1-\theta_1) + \left(\frac{1}{2} - \frac{\alpha_1}{3}\right)\theta_1, \\ &\frac{1}{a_2} - \frac{1}{3} = \frac{1}{6}(1-\theta_2) + \left(\frac{1}{2} - \frac{\alpha_2+1}{3}\right)\theta_2. \end{aligned} \tag{25}$$

Hence, by Lemma 1.6 and applying Gronwall's inequality and the energy inequality, we have

$$\begin{aligned} &\sup_{0\leq t\leq T}\left(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2\right) + \int_0^T \|Y_1^{\frac{\alpha_1}{2}}\nabla u\|_{L^2}^2 + \|Y_2^{\frac{\alpha_2}{2}}\nabla b\|_{L^2}^2\,dt \\ &\leq C\left(\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2\right), \end{aligned} \tag{26}$$

under the assumption of Theorem 1.2, which completes the proof.

### Acknowledgment

The research of the author was partially supported by the NSFC, China grant (No. 11801585).

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